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James Clerk Maxwell

THE DYNAMICAL THEORY OF THE ELECTROMAGNETIC FIELD

Edited by
THOMAS F. TORRANCE

James Clerk Maxwell

A Dynamical Theory of the Electromagnetic Field

with an appreciation by

ALBERT EINSTEIN

edited and introduced by

THOMAS F. TORRANCE

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A Dynamical Theory of the Electromagnetic Field
By Maxwell, James C. and Torrace, Thomas F.

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In Commemoration of

The Quatercentenary of the University of Edinburgh
1583–1983

and of

The Bicentenary of the Royal Society of Edinburgh
1783–1983

and in Homage to

James Clerk Maxwell
1831–1879

the greatest scientist to study in the former

and

the most distinguished fellow of the latter

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Preface

James Clerk Maxwell's essay *A Dynamical Theory of the Electromagnetic Field* was presented to the Royal Society in London on the 27th October, 1864, and was published in volume CLV of *Philosophical Transactions* in the following year. He was an unusually modest person who took his own achievements very lightly and was not given to speaking about them. His first biographers, however, Lewis Campbell and William Garnett, in their work *The Life of James Clerk Maxwell,* 1882, p. 342, have drawn attention to a letter which Clerk Maxwell wrote to a close relative rather excitedly about his new work. 'I have also a paper afloat, with an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns'. And so it proved to be, although it was so revolutionary in its conception, and moved so sharply away from scientific obsession with mechanical models and strictly mechanical explanations of natural phenomena, that it took some time for the scientific world to accept and appreciate it. The reaction of Clerk Maxwell's close friend to whom he was deeply indebted, Sir William Thomson (Lord Kelvin), was particularly interesting and enlightening. In an unpublished letter in the University of Glasgow, to which my attention has been drawn by Professor Sir John C. Gunn, 'Kelvin Papers M17', Clerk Maxwell refers to a charge made by Sir William that in his departure from a mechanical model of thought he had lapsed into *mysticism*! Thomson never really accepted this departure from the strict Newtonian tradition of mechanistic thought in spite of the verification of Clerk Maxwell's prediction of electromagnetic radiation in 1887, and even in the year 1904 when Einstein composed his four astonishing papers, Thompson still claimed that Clerk Maxwell's electromagnetic theory was not wholly tenable. But the onward march of physical science has ruled otherwise.

From his earliest days at Edinburgh Academy and Edinburgh University Clerk Maxwell had been fascinated with the relation of geometrical forms to motion, and developed new modes of thought, which he put very successfully into effect in several areas of scientific research and theory, in his explanation of the stability of Saturn's rings, in his dynamical theory of gases, in his work in colour vision and colour photography, and above all in his theoretical clarification of our understanding of electricity and magnetism and light through combining them in one electromagnetic theory. From his earliest studies, however, Clerk Maxwell also came to

realise the limited applicability of merely analytical mathematics to account for the dynamic modes of connection found in nature, so that even though he himself went further than any other between Newton and Einstein in the rigorous application of mathematical equations to natural phenomena and their behaviour, he was persistently aware of 'the vastness of nature and the narrowness of our symbolical sciences'. No human science, he felt, could ever really match up in its theoretical connections to the real modes of connection existing in nature, for valid as they may be in mathematical and symbolic systems, they were true only up to a point and could only be accepted by men of science, as well as by men of faith, in so far as they were allowed to point human scientific inquiry beyond its own limits to that hidden region where thought weds fact, and where the mental operation of the mathematician and the physical action of nature are seen in their true relation. That is to say, as Clerk Maxwell himself understood it, physical science cannot be rightly pursued without taking into account an all-important metaphysical reference to the ultimate ground of nature's origin in the Creator. Thus while Clerk Maxwell never intruded his theological, and deeply evangelical convictions, into his physical and theoretical science, he clearly allowed his Christian belief in God the Creator and Sustainer of the universe to exercise some regulative control in his judgment as to the appropriateness and tenability of his scientific theories, that is, as to whether they measured up as far as possible to 'the riches of creation'.

It was in that spirit that he put forward his own theories, always with reserve and always with the demand that they must be put to the test of fact, for his Christian faith would not allow him to fence off any area from critical clarification or to make any other claim for his theories than that they were of a provisional and revisable nature. That was characteristic of the scientific spirit which Clerk Maxwell found in Faraday, an account of which he took care to offer in concluding his article on *Faraday* for the *Encyclopaedia Britannica.* But what Clerk Maxwell said of Faraday there applies equally to himself.

A Dynamical Theory of the Electromagnetic Field was republished by W. D. Niven in the first volume of *The Scientific Papers of James Clerk Maxwell,* which he collected and edited in 1890. It is hoped that this new edition, in which it is now published for the first time as a separate work on its own, will make it more readily available. This is certainly needed, for, although it is one of the most important monographs in the whole history of science, it seems to be

rarely read and hardly ever known. No changes are made in the text, but I have taken the liberty of modernising some of the spelling and making the spelling consistent throughout. The occasional note in square brackets derives from W. D. Niven, the original editor.

Some of the terms used by Clerk Maxwell will sound rather strange to the modern reader, for the very changes in the conception of physical reality resulting from Clerk Maxwell's scientific work have affected the language that we now use. This applies in a rather subtle way to the terms 'force' and 'energy' and 'field', as the far-reaching implications of Michael Faraday's discoveries about moving lines of force and the physical reality of the field unfolded themselves to Clerk Maxwell's mathematical account of the intrinsic properties of the electromagnetic field. Clerk Maxwell himself made valiant attempts to modify his interpretation of fields of force in such a way that it would be consistent with classical mechanical theory, but it became steadily clearer that the concept of interacting force, together with the concept of the potential energy of a system, were deprived of their basis as the notion of instantaneous action at a distance had to be rejected. The effect of this change meant that the field, as determined by Clerk Maxwell's partial differential equations, now took the place of force. Thus we find him altering his terminology, not always consistently, to speak of 'the intrinsic energy of the electromagnetic field' as depending on electric and magnetic polarisation at every point, and put forward his general equations of the field as expressions of the relations between electric and magnetic fields and the rates of change in the fields with time and distance.

However, in attempting to offer an account of the development of Clerk Maxwell's theory of the electromagnetic field, faithful to Clerk Maxwell himself, I have retained the language he used at each stage of his advance, that is, in respect of *On Faraday's Lines of Force*, *On Physical Lines of Force*, and then *A Dynamical Theory of the Electromagnetic Field*. The modern reader will want to carry the change of terminology further; when instead of Clerk Maxwell's 'electromotive force' and 'magnetic force' he will want to speak of 'electric field vector' and 'magnetic field vector', and instead of 'electromagnetic momentum' he will speak of 'magnetic vector potential'. The reader should note that Clerk Maxwell has two uses for the terms 'electromagnetic force' and 'electromagnetic momentum', as they stand and as they are applied to circuits.

While Clerk Maxwell himself set out eight different equations in

this work, today the description 'Maxwell's equations' is generally restricted to four, as shown in vector notation.

1. $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$
2. $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \dot{\boldsymbol{D}}$
3. $\nabla \cdot \boldsymbol{B} = 0$
4. $\nabla \cdot \boldsymbol{D} = \rho$

Where $\boldsymbol{E}$ represents Electric Field Intensity; $\boldsymbol{H}$ Magnetic Field Intensity; $\boldsymbol{B}$ Magnetic Induction; $\boldsymbol{D}$ Electric Displacement; $\boldsymbol{J}$ Conduction Current Density; and ρ Charge Density.

Since Clerk Maxwell does not use vector quantities, though he does use triples of scalars, the reader will find it helpful to use a list giving translations of his triples into the symbols currently reserved for important physical quantities. Thus

electromagnetic momentum	$F, G, H \leftrightarrow$	vector potential $\boldsymbol{A}$
magnetic intensity	$\alpha, \beta, \gamma \leftrightarrow$	magnetic field $\boldsymbol{H}$
electromotive force	$P, Q, R \leftrightarrow$	electric field $\boldsymbol{E}$
current due to conduction	$p, q, r \leftrightarrow$	conduction current $\boldsymbol{j}$
electric displacement	$f, g, h \leftrightarrow$	electric displacement $\boldsymbol{D}$
total current	$p', q', r' \leftrightarrow$	total current $\boldsymbol{J}$
quantity of free electricity	$e \leftrightarrow$	charge density ρ
electric potential	$\Psi \leftrightarrow$	electric potential ϕ

I am happily indebted to Dr. A. D. Gilbert, of the Department of Mathematics in the University of Edinburgh, for careful reading of the proofs and for helpful suggestions in providing modern translations for Clerk Maxwell's terms and quantities. I thank him warmly. He points out that the equation (6) on p. 47 is only approximately true for $t \rightarrow \infty$, and that as it stands it is not completely accurate. He also draws attention to the fact that while for functions of position and time modern notation uses, for example, the partial derivative notation $\frac{\partial}{\partial t}$, Clerk Maxwell uses the total derivative notation $\frac{d}{dt}$ throughout. To be accurate by today's standards, it would be necessary to use partials at all appropriate places, principally from par. 55 p. 61 onwards.

James Clerk Maxwell must be allowed to speak for himself, but the appreciation of him by Albert Einstein, originally written for the year commemorating the centenary of his birth in 1831, must surely be recognised as the most authoritative guide there could be to Clerk Maxwell's significance in the history of science. A new

translation of that is offered here. This is included by the kind permission of Europa Verlag, Zürich, the publisher of Einstein's *Mein Weltbild*, edited by Carl Seelig, 1953. I am also indebted to the readiness of Crown Publishers, Inc., New York, in concurring with Europa Verlag in allowing me to present here a different English version of Einstein's tribute to Clerk Maxwell which they have long made available in Sonja Bargmann's translation in Albert Einstein, *Ideas and Opinions*, 1954.

Edinburgh
May, 1982

Thomas F. Torrance

Introduction

No one was ever more appreciative of Michael Faraday than James Clerk Maxwell. He found such a deep affinity with Faraday's physical insights and modes of thought that what Clerk Maxwell had to say of Faraday's place in the history of science throws not a little light on himself. Thus what Clerk Maxwell wrote of Michael Faraday shortly after his death in 1867 may be allowed to point us in the right direction in which to appreciate Clerk Maxwell's own place in the history of science, especially as it relates to the interpretation of the electromagnetic field.

Clerk Maxwell described Michael Faraday as 'the father of the enlarged science of electromagnetism which takes in at one view, all the phenomena which former inquirers had studied separately'.[1]

'He undertook no less a task than the investigation of the facts, the ideas, and the scientific terms of electromagnetism, and the result was the remodelling of the whole according to an entirely new scientific method . . .

'The high place which we assign to Faraday in electromagnetic science may appear to some inconsistent with the fact that electromagnetic science is an exact science, and that in some of its branches it had already assumed a mathematical form before the time of Faraday, whereas Faraday was not a professed mathematician, and in his writings we find none of these integrations of differential equations which are supposed to be of the very essence of exact science. Open Poisson and Ampère, who went before him, and you will find their pages full of symbols, not one of which Faraday would have understood. It is admitted that Faraday made some great discoveries, but if we put these aside, how can we rank his scientific method so high without disparaging the mathematics of these eminent men?

'It is true that no one can essentially cultivate any exact science without understanding the mathematics of that science. But we are not to suppose that the calculations and equations which mathematicians find so useful constitute the whole of mathematics. The calculus is but a part of mathematics.

'The geometry of position is an example of mathematical science established without the aid of a single calculation. Now Faraday's lines of force occupy the same position in electromagnetic science that pencils of lines do in the geometry of position. They furnish a method of building up an exact mental image of the thing we are

reasoning about. The way in which Faraday made use of his idea of lines of force in coordinating the phenomena of the magnetoelectric induction shows him to have been in reality a mathematician of a very high order—one from whom the mathematicians of the future may derive valuable and fertile methods.

'For the advance of the exact sciences depends upon the discovery and development of appropriate and exact ideas, by means of which we form a mental representation of the facts, sufficiently general, on the one hand, to stand for any particular case, and sufficiently exact, on the other, to warrant the deductions we may draw from them by the application of mathematical reasoning.

'From the straight line of Euclid to the lines of force of Faraday this has been the character of the ideas by which science has been advanced, and by the free use of dynamical as well as geometrical ideas we may hope for further advance. The use of mathematical calculations is to compare the results of the application of these ideas with our measurements of the quantities concerned in our experiments. Electrical science is now at the stage in which such measurements and calculations are of the greatest importance.

'We are probably ignorant even of the name of the science which will be developed out of the materials we are now collecting, when the great philosopher after Faraday makes his appearance.'[2]

In these brief paragraphs Clerk Maxwell put his finger upon features in Faraday's scientific activity which resounded to his own convictions, and quickened his combination of physical intuition with mathematical vision. From his natural bent and early training at Edinburgh Academy and Edinburgh University Clerk Maxwell was dissatisfied with merely analytical mathematics concerned with the manipulation of symbols detached from physical structures and patterns in nature, for the very point of mathematics was its bearing upon non-mathematical empirical reality.[3] Since he was especially concerned all through his life with dynamic states of matter he determined to develop forms of 'dynamical reasoning' in which to bring appropriate mathematical functions to bear upon the behaviour of nature in such a way as to disclose and interpret 'physical truth'. Outstanding examples of this kind of reasoning are to be found in his kinetic theory of gases, his development of the calculus of probabilities and his contribution to thermodynamics but above all in his dynamical theory of the electromagnetic field.

Clerk Maxwell was surely himself 'the great philosopher after Faraday' who took over the science that Faraday had handed on to

him, remodelling physics 'according to an entirely new method' while combining electricity, magnetism and light through a unifying theory in which he derived the structural field-laws of electromagnetic radiation. Through determining the mathematical properties of the unitary field in terms of a single constant, the ratio of the electromagnetic and electrostatic units of electricity, he was able to show that it was in fact equal to the constant of the medium which measured the speed of light.[4] When his theoretical prediction of electromagnetic waves was experimentally verified by Heinrich Hertz in 1887 Clerk Maxwell's equations were accepted by the scientific world, and the radical changes he had introduced into the epistemological and logical structure of physical science began to be appreciated for their epoch-making character. The way was now opened for the next great step forward in the unifying theories of Einstein. Who better than the great Max Planck, who discerned the profound relation between the thought of Clerk Maxwell and that of Albert Einstein, can indicate for us the place James Clerk Maxwell occupies in the history of modern science?

'While, in the kinetic theory of gases, Maxwell shared his leadership with several others, in the field of electrodynamics his genius stood alone. For to him was given, after many years of quiet investigation, a success which must be numbered among the greatest of all intellectual achievements. By pure reasoning he succeeded in wresting secrets from nature, some of which were only tested a full generation later, as a result of ingenious and laborious experiments. That such predictions are at all possible would be quite unintelligible if one did not assume that very close relations exist between the laws of nature and those of the mind.

'We must not of course forget that Maxwell did not build his Electrodynamic Theory in the air. For out of nothing comes nothing. He built his theoretical speculations on the firm foundations laid by the experimental work of Michael Faraday, whose memory we have so recently and fittingly celebrated. But Maxwell, with bold phantasy and mathematical insight, went far beyond Faraday, whose standpoint he both generalized and made more precise. He thus created a theory which not only could compete with the well established theories of electricity and magnetism but surpassed them entirely in success. For the criterion of the value of a theory, that it explains quite other phenomena besides those on which it is based, has never been so well satisfied as with Maxwell's theory. Neither Faraday nor Maxwell may have originally considered optics in

connection with their consideration of the fundamental laws of electromagnetism. Yet the whole field of optics, which had defied attack from the side of mechanics for more than a hundred years, was at one stroke conquered by Maxwell's Electrodynamic Theory; so much so that since then every optical phenomenon can be directly treated as an electromagnetic problem. This must remain for all time one of the greatest triumphs of human intellectual endeavour.'[5]

We must now consider Clerk Maxwell's scientific method for it was as unique as the new structure he gave to physical science and was indissolubly bound up with it. No scientist perhaps has demonstrated more clearly that method and subject-matter belong together and must be allowed constantly to interpenetrate each other if the mysteries of nature are to become naturally disclosed to our inquiries. As we have seen this was a distinctive feature of Faraday's scientific method which Clerk Maxwell singled out for appreciation, but Faraday's success showed to Clerk Maxwell that there had to take place a deeper intertwining of empirical and theoretical, or physical and mathematical, factors in the advance of physical science, and that is what he set out to achieve.

Later on, when he was no doubt reflecting on what he had been trying to do, he let fall some significant remarks about this in a review of *Elements of Natural Philosophy*, by Professors Sir William Thomson and P. G. Tait, in 1873.[6] He discussed the two main ways in which physical science or natural philosophy had long been taught. One method began with a thorough training in pure mathematics, so that when dynamical relations are afterwards presented to a student in the form of mathematical equations, he can appreciate the language, if not the ideas, of the new subject. 'The progress of science, according to this method, consists in bringing the different branches of science in succession under the power of the calculus.' The other method is to make the student familiar with physical phenomena, as well as the language of science, until he is able to perform and describe experiments of his own. 'Each of these types of men of science is of service in the great work of subduing the earth to our use, but neither of them can fully accomplish the still greater work of strengthening their reason and developing new powers of thought. The pure mathematician endeavours to transfer the actual effort of thought from the natural phenomena to the symbols of his equations, and the pure experimentalist is apt to spend so much of his mental energy on matters of detail and

calculation, that he has hardly any left for the higher forms of thought. Both of them are allowing themselves to acquire an unfruitful familiarity with the facts of nature, without taking advantage of the opportunity of awakening those powers of thought which each fresh revelation of nature is fitted to call forth.'

Clark Maxwell then went on to speak of 'a third method of cultivating physical science, in which each department in turn is regarded, not merely as a collection of facts to be coordinated by means of formulae laid up in store by the pure mathematicians, but as itself a new mathesis by which new ideas may be developed. Every science must have its fundamental ideas—modes of thought by which the process of our minds is brought into the most complete harmony with the process of nature—and these ideas have not attained their most perfect form as long as they are clothed with the imagery, not of the phenomena of the science itself, but of the machinery with which mathematicians have been accustomed to work problems about pure quantities.'

The point was reinforced at the end of the review. 'Mathematicians may flatter themselves that they possess new ideas which mere human language is yet unable to express. Let them make the effort to express these ideas in appropriate words without the aid of symbols, and if they succeed they will not only lay us laymen under a lasting obligation, but we venture to say, they will find themselves very much enlightened during the process, and will even be doubtful whether the ideas as expressed in symbols had ever quite found their way out of the equations of their minds.'[7]

It was through wrestling with Faraday's researches that Clerk Maxwell laid the basis for his *new mathesis* in which he sought to let the dynamic behaviour of nature in fields of force call forth from his mind the appropriate ideas through which it could be revealed and by which it could be given natural and rigorous interpretation. In the preface to his major work *A Treatise on Electricity and Magnetism* published in 1873, Clerk Maxwell tells us that he resolved to read no mathematics on the subject of electricity until he had read through Faraday's *Experimental Researches in Electricity*. 'I was aware that there was supposed to be a difference between Faraday's way of conceiving phenomena and that of the mathematicians, so that neither he nor they were satisfied with each other's language. I had also the conviction that the discrepancy did not arise from either party being wrong. . . As I proceeded with the study of Faraday, I perceived that his method of conceiving the phenomena

was also a mathematical one, though not exhibited in the conventional form of symbols. I also found that these methods were capable of being expressed in the ordinary mathematical forms, and thus compared with those of the professed mathematicians.

'For instance, Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance: Faraday saw a medium where they saw nothing but distance: Faraday sought the seat of the phenomena in real actions going on in the medium, they were satisfied that they had found it in a power of action at a distance impressed on the electric fluids.

'When I had translated what I considered to be Faraday's ideas into mathematical form, I found that in general the results of the two methods coincided, so that the same phenomena were accounted for, and the same laws of action deduced by both methods, but that Faraday's methods resembled those in which we begin with the whole and arrive at parts by analysis, while the ordinary mathematical methods were founded on the principle of beginning with the parts and building up the whole by synthesis.

'I also found that several of the most fertile methods of research discovered by the mathematicians could be expressed much better in terms of ideas derived from Faraday than in their original form.

'The whole theory, for instance, of the potential, considered as a quantity which satisfies a certain partial differential equation, belongs essentially to the method which I have called that of Faraday. According to the other method, the potential, if it is to be considered at all, must be regarded as the result of a summation of the electrified particles divided each by its distance from a given point. Hence many of the mathematical discoveries of Laplace, Poisson, Green and Gauss find their proper place in this treatise, and their appropriate expressions in terms of conceptions mainly derived from Faraday.'[9]

The conceptual instruments which Clerk Maxwell brought to bear on the electromagnetic phenomena disclosed through Faraday's researches enabling him to grasp and interpret their mathematical properties, were *real analogies* in nature and a way of thinking in *dynamic relations*.[10]

Under the conviction that real analogies permeate the regularities and harmonies of the creation across the whole spectrum of its manifestations, Clerk Maxwell used patterns of behaviour determined in one set of natural phenomena to assist him in imagining

and working out the patterns inherent in another set of phenomena. Thus, for example, he developed the analogy between the rotatory movement of a fluid and the moving lines of force in an electromagnetic field, or the analogy between vibrations in an elastic medium and the behaviour of waves of light. The value of analogical thinking of this kind is that it gave Clerk Maxwell 'an extraordinary flexibility in his researches', for it enabled him constantly to develop provisional models and theories in the hard work of seeking to match thought with reality in such a way as to yield appropriate ideas. Moreover, as William Berkson has expressed it, 'It encouraged him to propose theories which he himself found very implausible, but which gave him mathematical systems which were very enlightening when applied to the phenomena he was trying to learn about. The liberating feature of the 'method of analogies' is the fact that it encouraged the development of what one believes to be *false* analogies for the light they may shed on the truth—a procedure which had been recommended by Faraday. The method also encourages an individual to construct theories other than his pet ones, and in general encourages the invention of theories which *nobody* would believe for their possible fruitfulness.'[11]

Relational thinking derives from the realisation that in its natural state the universe is characterised by inherent integrations and continuities, so that the relations between things belong to what things really are. This means that if we are to understand physical realities we must begin not from a consideration of individual particles into which they may be analysed and then considered in their relations to one another, but from their original interrelations, and from the integrated patterns they exhibit seek to understand their parts and internal relations. This way of proceeding from the whole to the parts instead of from the parts to the whole seemed to be the method of Faraday. In Clerk Maxwell's words: 'He never considers bodies as existing with nothing between them but their distance, and acting on one another according to some function of that distance. He conceives all space as a field of force, the lines of force being in general curved, and those due to any body extending from it on all sides, their directions being modified by the presence of other bodies. He even speaks of the lines of force belonging to a body as in some sense part of itself, so that in its action on distant bodies it cannot be said to act where it is not. This, however, is not a dominant idea with Faraday. I think he would rather have said that the field of space is full of lines of force, whose arrangement

depends on that of bodies in the field, and that the mechanical and electrical action on each body is determined by the lines which abut on it.'[12]

Clerk Maxwell was evidently concerned to make the point that while Faraday could speak of the interrelations of bodies in a field of force as contributing to what those bodies really are, which would make for a notion of a *continuous* field, what he generally had in mind was that throughout space bodies interact mechanically and electrically with one another, not instantaneously at a distance, but at neighbourly points, which would make for the notion of a *contiguous* field. That was certainly the conception that Clerk Maxwell took over from Faraday when he tried to determine the laws of the force field in accordance with the requirements of Newtonian laws which apply to bodies in their external relations with one another, but his dynamical and relational mode of thinking gave rise to mathematical equations and an interpretation of the field which led him beyond Faraday to a notion of physical reality represented by continuous fields. That was the far-reaching implication of this work on a dynamical theory of the electromagnetic field, which represents a decisive break in the mechanistic interpretation of the universe that had prevailed since Newton.

Clerk Maxwell's progress to this point in his physical science registered several stages which were marked by distinctive publications in which he committed himself ever more deeply to the concept of the field and struggled to clarify its immanent mathematical and dynamical properties.

In his first work, *On Faraday's Lines of Force,* published in 1855,[13] he surveyed the ground and through analogical reasoning gave Faraday's physical insights into electromagnetic phenomena some kind of mathematical order without as yet developing an appropriate theory. He set himself to do two things, to find a physical analogy which would help the mind to grasp the results of all previous investigations into electricity and magnetism, and to show that a development of Faraday's concept of moving lines of force would not be inconsistent with the mathematical formulae of Poisson or the laws established by Ampère which did not conflict with Newtonian mechanics.

We cannot do better than repeat Clerk Maxwell's own account of his intentions. 'The first process in the effectual study of the science, must be one of simplification and reduction of the results of previous investigations to a form in which the mind can grasp them.

The results of this simplification may take the form of a purely mathematical formula or of a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connections of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.

'In order to obtain physical ideas without adopting a physical theory we must make ourselves familiar with the existence of physical analogies. By a physical analogy I mean that partial similarity between the laws of one science and those of another which makes each of them illustrate the other. Thus all the mathematical sciences are founded on relations between physical laws and laws of numbers, so that the aim of exact science is to reduce the problems of nature to the determination of quantities by operations with numbers. Passing from the most universal of all analogies to a very partial one, we find the same resemblance in mathematical form between two different phenomena giving rise to a physical theory of light.

'The changes of direction which light undergoes in passing from one medium to another, are identical with the deviations of the path of a particle in moving through a narrow space in which intense forces act. This analogy, which extends only to the direction, and not to the velocity of motion, was long believed to be the true explanation of the refraction of light; and we still find it useful in the solution of certain problems, in which we employ it without danger, as an artificial method. The other analogy, between light and the vibrations of an elastic medium, extends much farther, but, though its importance and usefulness cannot be over-estimated, we must recollect that it is founded only on a resemblance *in form* between the laws of light and those of vibrations. By stripping it of its physical dress and reducing it to a theory of "transverse alternations" we might obtain a system of truth strictly founded on observation, but probably deficient both in the vividness of its conceptions and the fertility of its method. . .'[14]

This reference to light was typical of the procedure which Clerk Maxwell followed throughout the paper. In Part I he made extensive use of an analogy taken from the motion of an incompressible fluid. By drawing out a relation between the lines of flow of a fluid of this kind and the lines of force which govern the behaviour of the electromagnetic field, he sought to determine not merely the laws of statical electricity in a single medium but a way of representing the patterns of behaviour when action passes from one dielectric to another. However, Clerk Maxwell insisted that the fluid in question was not even a hypothetical fluid but 'merely a collection of imaginary properties which may be employed for establishing certain theorems in pure mathematics in a way more intelligible to many minds and more applicable to physical problems than that in which algebraic symbols alone are used'.[15] In Part II of the paper Clerk Maxwell turned to consider the idea of 'the electro-tonic state', i.e. the special state characterising space traversed by lines of magnetic force. By a careful study of the laws of elastic solids and of the motions of viscous fluids, Clerk Maxwell hoped to discover a method of forming 'a mechanical conception of the electro-tonic state adapted to general reasoning'. He admitted, however, that the idea of the electro-tonic state had not yet presented itself to his mind in such a form that its nature and properties might be clearly explained 'without reference to mere symbols'. Hence after developing a series of mathematical theorems and relevant equations to represent the electro-tonic state he remained dissatisfied. 'We may conceive of the electro-tonic state at any point of space as a quantity determinate in magnitude and direction, and we may represent the electro-tonic condition of a portion of space by any mechanical system which has at every point some quantity, which may be a velocity, a displacement, or a force, whose direction and magnitude correspond to those of the supposed electro-tonic state. This representation involves no physical theory, it is only a kind of artificial notation.'[16]

'What is the use then of imagining an electro-tonic state of which we have no distinctly physical conception, instead of a formula of attraction which we can readily understand? I would answer, that it is a good thing to have two ways of looking at a subject, and to admit that there *are* two ways of looking at it. Besides, I do not think that we have any right at present to understand the action of electricity, and I hold that the chief merit of a temporary theory is, that it shall guide experiment, without impeding the progress of the true theory when it appears.'[17]

It is difficult to avoid the conclusion that in studying Faraday's researches Clerk Maxwell had already formed a powerful intuition of the nature of the electromagnetic field, and that his artificial extension of real analogies and his design of rather implausible mechanical models were meant to meet the dictates of his prior intuitive grasp of how the field would behave. But the quite imaginary character of analogical patterns and the mechanical impossibility of his models were bound to suggest that the true nature of the field when it became established would transcend mechanical representations of it. Meantime, however, Clerk Maxwell justified his attempts at mechanical explanation by appealing to the advantage of having two ways of looking at things. And there could be no doubt that so long as some sort of ether was presumed in the behaviour of electromagnetic phenomena, mechanical properties could not be excluded from the concept of the field, but Faraday's concept of pervasive lines of force throughout space had the effect of putting a serious question-mark to ether as a serious scientific idea. Faraday's scientific ideas and physical hypotheses seemed to Clerk Maxwell to be so fundamentally 'natural' that he had a profound respect for Faraday's instinct. In a reference to Faraday's notion of the electro-tonic state of space traversed by magnetic lines of force in face of the questionable nature of empirical evidence supporting it, Clerk Maxwell gave him his vote. 'The conjecture of a philosopher', he said with reference to Faraday, 'so familiar with nature may sometimes be more pregnant with truth than the best established experimental law discovered by empirical inquirers, and though not bound to admit it as a physical truth, we may accept it as a new idea by which our mathematical conceptions may be rendered clearer.'[18] That was precisely how Clerk Maxwell himself proceeded, impelled by intuitive insight and scientific instinct to develop a genuine theory of the electromagnetic field.

The next stage in Clerk Maxwell's thought is represented by the publication of *On Physical Lines of Force* in volume xxi of the *Philosophical Magazine* in 1861/2.[19] While he had contented himself in writing *On Faraday's Lines of Force* with reducing Faraday's ideas as far as possible to geometrical form, in this new work Clerk Maxwell introduced basic theoretical principles which began to set everything on a new basis, but he also worked with another elaborate model in order, as he said, 'to examine magnetic phenomena from a mechanical point of view, and to determine what tensions in, or motions of, a medium are capable of producing the mechanical phenomena observed. If, by the same hypothesis, we can connect

the phenomena of magnetic attraction with electromagnetic phenomena and with those of induced currents, we shall have found a theory which, if not true, can only be proved to be erroneous by experiments which will greatly enlarge our knowledge of physics'.[20] What Clerk Maxwell wanted was a model of the magnetic field which would illustrate Faraday's law of magnetic induction, but in his hands the model also served a heuristic purpose, for it was designed as a way of interrogating nature itself in order to obtain true solutions to questions which his mathematical theory, and in this instance his demand for symmetry, suggested.[21] This is what actually happened, for, as Sir J. J. Thomson expressed it, 'when he came to use the model he found that it suggested that *changes* in the electric force would produce magnetic force'.[22] When Clerk Maxwell generalised these ideas, that a changing electric field produces a magnetic field, and a changing magnetic field produces an electric field, and gave them rigorous mathematical expression, the result was his equations of the field which tell how electricity and magnetism propagate themselves in accordance with exact laws. Hence Sir J. J. Thomson claimed that the introduction and development of this theory was Clerk Maxwell's greatest contribution to physics.

The actual model which Clerk Maxwell thought up was extremely ingenious, intricate but not really credible, for he had to keep on modifying it by the introduction of his famous 'idle wheels' and the concept of an 'elastic medium' in a way that removed it from practical mechanics, but, as we have just indicated, it served his purpose rather well.[23] This was a model in which he imagined the magnetic field being occupied by molecular vortices the axes of which coincided with the lines of force. But it is best to let Clerk Maxwell himself summarise what he wrote, as he did in Part III of his work.

'In the first part of this paper I have shown how the forces acting between magnets, electric currents, and matter capable of magnetic induction may be accounted for on the hypothesis of the magnetic field being occupied with innumerable vortices of revolving matter, their axes coinciding with the direction of the magnetic force at every point in the field.

'The centrifugal force of these vortices produces pressures distributed in such a way that the final effect is a force identical in direction and magnitude with that which we observe.

'In the second part I described the mechanism by which these rotations may be made to coexist, and to be distributed according to the known laws of magnetic lines of force.

'I conceived the rotating matter to be the substance of certain cells, divided from each other by cell-walls composed of particles which are very small compared with the cells, and that it is by the motions of these particles, and their tangential action of the substance in the cells, that the rotation is communicated from one cell to another.

'I have not attempted to explain this tangential action, but it is necssary to suppose, in order to account for the transmission of rotation from the exterior to the interior parts of each cell, that the substance in the cells possesses elasticity of figure, similar in kind, though different in degree, to that observed in solid bodies. The undulatory theory of light requires us to admit this kind of elasticity in the luminiferous medium, in order to account for transverse vibrations. We need not then be surprised if the magnetoelectric medium possesses the same property.

'According to our theory, the particles which form the partitions between the cells constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the cells is electromotive force, and the pressure of the particles on each other corresponds to the tension or potential of the electricity.

'If we can now explain the condition of a body with respect to the surrounding medium when it is said to be "charged" with electricity, and account for the forces acting between electrified bodies, we shall have established a connection between all the principal phenomena of electrical science.'[24]

Clerk Maxwell granted frankly the implausibility of the model and did not want to be thought of as in any way identifying its operation with what one finds in nature. 'The conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connection existing in nature, or even as that which I would willingly assent to as an electrical hypothesis. It is, however, a mode of connection which is mechanically conceivable, and easily investigated, and it serves to bring out the actual mechanical connections between the known electromagnetic phenomena; so that I venture to say that any one who understands the provisional and temporary character of this hypothesis, will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena.'[25] This was a point which Clerk Maxwell took care to reinforce, in the publication of *A Treatise on Electricity and Magnetism* in 1873. 'The attempt which I then made to imagine a working

model of this mechanism must be taken for no more than it really is, a demonstration that mechanism may be imagined capable of producing a connection mechanically equivalent to the actual connection of the parts of the Electromagnetic Field.'[26]

The astonishing thing is that with the assistance of this strange imaginary model Clerk Maxwell was able to derive the partial differential equations, for which he is so famous, to describe the relations between electric currents and magnetic fields and thus to give us the structural laws of the electromagnetic field. As William Berkson has summarised it, Clerk Maxwell derived his equations in three stages. 'First, he used the assumption of the vortices to account for purely magnetic effects. Second, he used the assumption of the electrical balls to derive the relations between current and magnetism, including induction. Third, he used the assumption of elasticity of the electrical balls to account for the effects of static charge. Each of these stages were steps towards Maxwell's crowning achievement: the electromagnetic theory of light.'[27]

That is to say, the solution of Clerk Maxwell's equations suggested the existence of electromagnetic waves travelling with the speed already established for light. 'The velocity of transverse undulations in our hypothetical medium, calculated from the electromagnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from optical instruments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena*.'[28] Thus Clerk Maxwell achieved a unified theory of electricity, magnetism and light in which the electromagnetic and luminiferous media or fields were the same. In this achievement, however, his partial differential equations had the effect not only of confirming Faraday's idea that vibrations without a vibrating matter, lines of force moving in empty space, are real, but also of establishing the *independent reality* of the electromagnetic field.[29] This also had the effect of altering the concept of the field as a medium of interaction between contiguous forces and of demanding its reconsideration. Moreover, since it became clear that Clerk Maxwell's theory of the electromagnetic field was only loosely and provisionally related to his mechanical model, the basic question had to be raised about the propriety of a mechanical approach to the interpretation of electromagnetic phenomena.

This brings us to the third stage in the progress of Clerk Max-

well's understanding of the electromagnetic field, in a paper presented to the Royal Society in London in 1864, *A Dynamical Theory of the Electromagnetic Field*, now republished here.[30] According to W. D. Niven, his former student and the editor of his *Scientific Papers*, this 'great memoir' contains Clerk Maxwell's mature thoughts on the subject.[31] What is distinctive about this paper, as we shall see, is that he embodied his partial differential equations in a thoroughly *relational* interpretation of the electromagnetic field without reliance on or recourse to the mechanical model which he had invented in *On Physical Lines of Force*, and indeed in marked independence of Newtonian mechanics.

Clerk Maxwell himself tells us that the theory he now proposed, in deliberate contrast to the mechanical theories of Weber and Neumann, may be called 'a theory of the *Electromagnetic Field*, because it has to do with the space in the neighbourhood of the electric or magnetic bodies, and it may be called a *Dynamical* Theory, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced'.[32] In this theory he assumed the reality of a medium internally connected in such a way that the motion of one part depends in some way on the motion of the rest, and that these connections are capable of a certain kind of elastic yielding, 'since the communication of motion is not instantaneous, but occupies time'.[33] 'The medium is therefore capable of receiving and storing up two kinds of energy, namely, the "actual" energy depending on the motion of its parts, and "potential" energy, consisting of the work which the medium will do in recovering from displacement in virtue of its elasticity. The propagation of undulations consists in the continual transformation of one of these forms of energy into the other alternately, and at any instant the amount of energy in the whole medium is equally divided, so that half is energy of motion, and half is elastic resilence.' 'A medium having such a constitution', Clerk Maxwell added, 'may be capable of other kinds of motion and displacement than those which produce the phenomena of light and heat, and some of these may be of such a kind that they may be evidenced to our senses by the phenomena they produce.'[34]

The procedure Clerk Maxwell followed was the reverse of that traditionally accepted in the nineteenth century, the derivation of basic theoretical concepts by deducing them from the analytical particulars of observations. Instead he brought forward intuitively reached concepts suggested to him through his familiarity with the

behaviour of nature and then deduced their consequences for physical reality and the kind of knowledge we may build up of it on mechanical assumptions. That is to say, the theories put forward in this way had to be tested in respect of the range of empirical facts for which they could account and in respect of their capacity both to unify knowledge already gained and to bring to light new properties in nature demanding interpretation. As Clerk Maxwell himself regarded it, this was basically the method of Faraday, except that it gave greater precision to mathematical form without, however, impairing the natural integration of empirical and theoretical ingredients in knowledge or in the physical and dynamical reasoning so characteristic of Faraday.[35] Thus instead of seeking to determine the laws of the electromagnetic field through abstractive and logical processes from mechanical principles and deducing the behaviour of electricity and magnetism, Clerk Maxwell put forward laws of induction and then deductively tested them by determining their consistency with known facts and well established theories and laws.

Hence, after recalling the general phenomena of the interaction of currents and magnets, the induction produced in a circuit by variations in the strengths of the field, the properties of dielectrics and so on, Clerk Maxwell abruptly introduced 'the General Equations of the Electromagnetic Field', but there is no suggestion that they have any dependence on a definite mechanical model. The significant point to note is that these general equations are presented as the mathematical expression of the *intrinsic energy* of the electromagnetic field.[36] It is through their application to various properties and features of the field that a theory of the field is to be formulated. Thus when they are applied, for example, to magnetic disturbances propagated through a non-conducting field, it can be shown that the only disturbances which can be so propagated are those which are transverse to the direction of propagation, and that the velocity of propagation is the velocity v, found from experiments such as those of Weber, which expresses the *constant* number of electrostatic units of electricity which are contained in one electromagnetic unit now known as c. 'This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws. . .'[37]

In the course of his argument throughout the paper and in the application of these general equations to the electromagnetic field,

Clerk Maxwell finds himself using 'mechanical' language to describe the behaviour of phenomena in the field. Then he pulls himself up, and adds two significant paragraphs.

'I have on a former occasion attempted to describe a particular kind of strain, so arranged as to account for the phenomena.[38] In the present paper I avoid any hypothesis of this kind; and in using such words as electric momentum and electric elasticity in reference to known phenomena of the induction of currents and the polarization of dielectrics, I wish merely to direct the mind of the reader to mechanical phenomena which will assist him in understanding the electrical ones. All such phrases in the present paper are to be considered as *illustrative, not as explanatory*.

'In speaking of the energy of the field, however, I wish to be understood *literally*. All energy is the same as mechanical energy, whether it exists in the form of motion or in that of elasticity or in any other form. The energy in electromagnetic phenomena is mechanical energy. The only question is, Where does it reside? On the old theories it resides in the electrical bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves, and is in two different forms, which may be described without hypothesis as magnetic polarization and electric polarization, or, according to a very probable hypothesis, as the motion and the strain of one and the same medium.'[39]

By linking his partial differential equations with the intrinsic energy of the field as their mathematical expression, and not, directly at any rate, with a mechanical interconnection of bodies in motion, Clerk Maxwell found that electric and magnetic fields could exist in free or empty space quite independent of material bodies, and so finally established the independent existence and reality of the electromagnetic field, which required an alteration in the concept of the field. Moreover, by showing that these equations enabled him to derive the laws of electromagnetic radiation and electromagnetic fields at any point in space and time, he established the continuous nature of the field in contrast to Newtonian and Cartesian concepts of the field in terms of the action of bodies or particles externally on one another either at a distance or at contiguous points.[40]

This implies that in the final analysis the concept of ether which

demands and underlies a mechanical understanding of electromagnetic phenomena is otiose. Here, however, there appears to be an ambiguity in Clerk Maxwell's thought, for he evidently did not quite shake himself clear of the notion of ether,[41] or therefore of the need to offer descriptions of phenomena in terms of mechanical relations and models, even though he insisted, as we have seen, that the words and phrases he used in these descriptions must be taken in an illustrative and not in an explanatory sense. The reason for the ambiguity would seem to have to do with his idea that while all energy is one, it manifests itself in two forms, as potential and as kinetic energy. Elsewhere Clerk Maxwell could insist that we have to reckon with two ways of looking at things, so that we must learn to screw up the telescope of theory sometimes to one pitch of definition, and sometimes to another, if we are really to see down into the depths of things—otherwise everything merges dimly together.[42] In other words, nature, as we seek to apprehend it through our scientific theories manifests itself to us at different levels, one where we are concerned with subordinate, mechanical connections, and another where we are concerned with real continuous connections. Regarded in this way, we would have to interpret Clerk Maxwell's physical science as concerned with two different levels of connection, one which is somewhat arbitrary and artificial which is the limiting case of the other where we have in view real connections inherent in the phenomena of nature. This would explain why in this work he is concerned to deduce what he called 'mechanical actions in the field' from his fundamental explanations of the dynamic relations of the electromagnetic field in terms of equations which are expressive of the mathematical properties of its intrinsic and pervasive energy, but which are not themselves mechanically explicable.[43] In so far as the non-mechanical laws of the continuous field can be shown to be consistent, within defined limits, with the laws governing the mechanical interaction of bodies separate from one another, they cannot but carry added conviction.

The most famous section of *A Dynamical Theory of the Electromagnetic Field*, of course, is that in which Clerk Maxwell set forth his 'Electrodynamic Theory of Light'.[44] This was surely the peak in the progression of his thought, at which he demonstrated that a theory of light can be derived directly from his electromagnetic equations. He had already laid the foundation for this achievement in his earlier work *On Physical Lines of Force*, as we have seen, but

now with a less sophisticated and certainly a very elegant set of equations he showed that the velocity of electromagnetic waves and the velocity of light both have the same value v in a vacuum as the constant ratio of the electromagnetic and electrostatic units of electricity, known as c. That was quite decisive. '*The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.*'[45]

Moreover, since Clerk Maxwell's equations show how electricity and magnetism behave under conditions of space and time they made possible the deduction of new properties of electromagnetic waves and suggested the existence of other wave-like perturbations which are propagated from place to place with the same finite velocity beyond those which give rise to visible phenomena, that is, perturbations in which the electric intensity of the field would have to vary with extreme rapidity, for example, a thousand million million oscillations a second (10^{14} Hz). That prediction, as we have already had occasion to recall, was verified very ingeniously by Heinrich Hertz when he found waves of a very high frequency which were propagated with the speed of light waves beyond those which we speak of as 'visible'.[46] The fact that the basic deductions from his equations which Clerk Maxwell carried out in the latter part of his work proved to be correct, showed that they promised to be the ground of new discoveries hitherto undreamed of. In fact, the use of Clerk Maxwell's equations involves such a close integration between empirical and theoretical elements in scientific knowledge that Einstein could speak of them as 'the natural expression of the primary realities of physics'.[47]

Eight years after he wrote *A Dynamical Theory of the Electromagnetic Field,* Clerk Maxwell published a very much larger work entitled *A Treatise on Electricity and Magnetism,* in 1873. Nearly all of the former work, apart from a few insignificant details, was incorporated in this work, while it was his radically dynamical and relational account of electricity and magnetism that characterised his treatment.[48] In the first edition of this treatise he tells us that he proposed to describe the most important electrical and magnetic phenomena, to show how they may be subjected to measurement and to trace the mathematical connections of the quantities measured. 'Having thus obtained the data for a mathematical theory of electromagnetism and having shown how this theory may be applied to the calculation of phenomena, I shall endeavour to place in as

clear a light as I can the relations between the mathematical form of this theory and that of the fundamental science of Dynamics, in order that we may be in some degree prepared to determine, the kind of dynamical phenomena among which we are to look for illustrations or explanations of the electromagnetic phenomena.'[49]

The fact that he could apparently speak of 'illustrations' and 'explanations' as somehow parallel would seem to indicate that he was withdrawing his earlier reserve about mechanical explanation, but that would hardly be justified, for far from going back on the point that his general equations and the laws of the field which they express are independent of mechanical models, he exploits that independence even further. However, what Clerk Maxwell does do in the Treatise is to build round the essential exposition of his theory a formidable scaffolding of a mechanical and dynamical kind which demonstrates that the theory of electromagnetic radiation which he has advanced and now elaborates in great detail is thoroughly consistent with the established structure of physics in accordance with classical Newtonian mechanics. This is not to accept 'action at a distance' notions of causality or the prescriptive imposition upon the behaviour of the world of absolute uniform notions of space and time, and thus a rigidly molecularist and determinist view of nature, but rather to suggest that, when we probe more deeply into the properties of the intrinsic energy that pervades the universe, we come up with laws valid at that level which need not conflict with laws that hold good at another level where we are not concerned with relations that are continuous in space and time. This two-fold way of looking at things, and a corresponding duality in method, are emphasised by Clerk Maxwell from the very start in his preface. What he himself is primarily intent on pursuing, however, is the study of electromagnetic phenomena in terms of their internal relations in the fields of force in which they are found and the development as far as possible of a mathematical account of them in a methodical manner. That is to say, Clerk Maxwell, like Faraday, sought the seat of electromagnetic phenomena in the real actions going on in the medium or the continuum in which the lines of force belonging to bodies are in some sense part of them.[50]

In *A Treatise on Electricity and Magnetism* Clerk Maxwell drew a sharper distinction between mechanical force and electromotive force. 'Electromotive force is always to be understood to act on electricity only, not on the bodies in which electricity resides. It is never to be confounded with ordinary mechanical force, which acts

on bodies only, and not on the electricity in them.'[51] This distinction falls into line with the two kinds of energy he spoke of in *A Dynamical Theory of the Electromagnetic Field.* Granted that all energy is the same in kind, it nevertheless differs in form, with respect to *motion* and *position*, so that we must seek to understand energy in a two-fold way, through the dynamics of fields and the mechanics of bodies in interaction.[52] When we have to do with the latter, integral and differential equations are the appropriate mathematical expression, i.e. for a theory of action between particles at a distance, but when we have to do with the former, partial differential equations are the appropriate mathematical expression.[53]

If Clerk Maxwell's thought, especially as set forth in *A Treatise on Electricity and Magnetism*, is to be interpreted in this way, then it becomes easier to understand why he persisted with holding on to the mechanical scaffolding of his electromagnetic theory even when he had succeeded in penetrating more deeply than ever before into the understanding of connections in nature in terms of continuous fields and their laws. So far as the use of, or reliance on, mechanical models is concerned, Clerk Maxwell's theory of electromagnetism and light broke quite free from them, for the general equations from which his theory is derived are quite independent of mechanical structure. Mechanical models may continue to be used in an illustrative way to assist the mind by giving it something distinct to lay hold of, but once they have fulfilled their role they must be kicked away. So far as the mechanical scaffolding is concerned, however, there is no doubt that it must be allowed to remain, if we are to recognise that while all energy is one it manifests itself in two forms in respect of the position and behaviour of discrete particles and in respect of the spatial and temporal characteristics of the field. In this event, however, real modifications may have to be introduced into the mechanical description of the world in accordance with the principles of classical Newtonian science, by way of deducing and regaining mechanical concepts in accordance with the principles which physical science acquires from the new perspective governed by field theories and laws. It looks as if that was *also* what Clerk Maxwell was intent on doing in this later work especially.

Nevertheless, there can be no doubt that Clerk Maxwell remained ill at ease with his own work right up to the end of his life, for somehow his theories did not satisfy his standards of tenability. There is no evidence that he was dissatisfied with his general equations of the electromagnetic field or his electromagnetic theory

of light, but they needed to be interpreted through a more natural embodiment in created reality than they were given in his elaborate mathematicisation. In this respect Clerk Maxwell's last unfinished work is of some interest, *An Elementary Treatise on Electricity*, which was completed and published by W. Garnett in 1881, two years after Clerk Maxwell's Death. Here are several sentences from the preface.

'In this smaller work I have endeavoured to present, in as compact a form as I can, those phenomena which appear to throw light on the theory of electricity and to use them, each in its place, for the development of electrical ideas in the mind of the reader. In the larger treatise I sometimes made use of methods which I do not think the best in themselves, but without which the student cannot follow the investigations of the founders of the Mathematical Theory of Electricity. I have since become more convinced of the superiority of methods akin to those of Faraday, and have therefore adopted them from the first.'[54]

This would seem to indicate that Clerk Maxwell felt that his theory of the electromagnetic field lacked physical interpretation and required to be embodied in the empirical realities of electromagnetic phenomena if they were to be appreciated—but the fact that he more or less abandoned it three or four years before he died prematurely in 1879, or at least that he added little to the first eight chapters during that period, may reflect his conviction that his own accounts needed the kind of integration with physical truth which he had learned to appreciate in Faraday's work.[55] In these extant chapters he did little more than describe and demonstrate the principal facts relating to electromagnetic phenomena and offer some deductions from them, which corresponded in a more elementary way, as it was intended, to the first part of *A Treatise on Electricity and Magnetism*, so that he did not get down to the real substance of what he wanted to do in showing the geometric embodiment of his theory in established empirical facts.

If William Berkson is right in his interpretation, Faraday, who clearly rejected the notion of ether, operated with a much closer integration of matter, force and field than Clerk Maxwell, for in the last resort he seems to have regarded particles as converging lines of force, and even as modes of force themselves.[56] Hence Faraday regarded force as acting directly on force at contiguous points in the field, and all space was held to be a vast field of the interaction of various forces. According to Berkson, then, the identification of

matter and field was the key step which Faraday took, whereas for Clerk Maxwell matter and field were separate though interpenetrating entities.[57] In view of this it may well be the case that behind Clerk Maxwell's unease with his attempts to surround his theory with a mechanical scaffolding of some kind lay the intuition that his own general equations which expressed the intrinsic energy of the continuous field matched Faraday's concepts of matter and force better than his own, and that it was his hope to be able to integrate his equations and Faraday's understanding of matter and force in the dynamic structure of the field. Certainly Clerk Maxwell's last book in the fragmented from in which it is extant does not measure up to this, although, had Clerk Maxwell not been so heavily involved in his last years with editing the Cavendish papers, he might have had time to carry through his thought in this direction with his remarkably fresh and creative power. That would have taken him right on to the brink of relativity theory! Certainly Clerk Maxwell's unification of electricity, magnetism and light through the power of his partial differential equations provided Einstein with the basis on which, and in the light of the Lorentz transformations, he was able to crystallize his theory of relativity.[58]

Three observations may now be offered in concluding this Introduction.

(1) Clerk Maxwell 'created for the first time a field theory which was independently testable against Newtonian force theories'.[59] He created a situation in which the dominance of Newtonian mechanics over the whole spectrum of physical science was called into question and decisive steps were taken in the direction of a non-mechanical thoroughly relational understanding of the intelligible connections immanent in the universe. No doubt Clerk Maxwell did not realise the far-reaching implications of his work which was to change the basic perspective and direction of physical science and alter our understanding of the world of space and time. With reference to Clerk Maxwell's two basic papers, *On Physical Lines of Force* and *A Dynamical Theory of the Electromagnetic Field*, Ivan Tolstoy has recently given us the following appraisal. 'For us, with our hundred or so years of perspective, these two papers—Maxwell's theory of electromagnetism—are a turning point in the history of science. The theory is, first of all a synthesis—one of the greatest in the history of science. It unifies two kinds of force—the electric and the magnetic—under one: the electromagnetic field. This unification was the direct, logical consequence of Faraday's experimental work; it had

been begun by others—Ampère, Weber, W. Thomson. But Maxwell crystallized this, the first of the modern unified field theories and gave it the mathematical form which remains immortal under the name of *Maxwell's equations*—a system of relationships between changing electric and magnetic fields—a whole universe of electromagnetic phenomena, miraculously contained in a few lines of elegant mathematics.'[60]

(2) Clerk Maxwell's work was of profound conceptual importance for it had the effect of reorganising the epistemological and logical substructure of physical science, not only through his determination of the mathematical properties of radiation which has had immense implications for scientific technology, but through the way in which he conceived and developed the nature of the field and established the reality of the field as the underlying reality of all spatio-temporal phenomena. At this point we cannot do better than let Einstein himself speak. 'The formulation of these equations is the most important event in physics since Newton's time, not only because of their wealth of content, but also because they form a pattern for a new type of law. The characteristic features of Maxwell's equations, appearing in all other equations of modern physics, are summarized in one sentence. Maxwell's equations are laws representing the *structure* of the field. . . . All space is the scene of these laws and not, as for mechanical laws, only points in which matter or charges are present.'[61] Moreover, it should be pointed out, that since the epistemological form of Clerk Maxwell's general equations does not depend on the way in which the observer, or the person who measures the fields, is moving,[62] they have the effect of establishing the objectivity of scientific knowledge in a new and a profounder way than was possible in the post-Newtonian, and certainly, the post-Kantian, outlook upon the universe.

(3) Physical science as it stemmed from Clerk Maxwell's revolutionary ideas was left with a serious, and perhaps an ultimately irresolvable problem, of which, as we have seen, he himself seems to have been aware. This relates to the fact that although his equations expressed the mathematical properties of the energy intrinsic to the continuous field of space and time, he was unable to reconcile in a satisfactory manner, the ways in which the two basic forms of this energy, in respect of position and motion, manifest themselves. Thus, as Einstein has expressed it, while Clerk Maxwell's partial differential equations appeared as the natural expression of the primary realities of physics, in a particular area of theoretical

physics, 'the continuous field appeared side by side with the material point as the representative of physical reality. This dualism has to this day not disappeared, disturbing as it must evidently be to any systematic mind.'[63] Clerk Maxwell's problem remains with us in the difficulties that have emerged in the reconciliation of relativity theory and quantum theory, not to mention a unified field theory which will take in thermodynamics and gravity theory as well—although some way toward the solution may well lie along the line of thought which both Michael Faraday and Clerk Maxwell entertained, that the relations between particles in a field of force must be thought of as constituting, in part at least, what particles actually are. However, it is doubtful whether in the nature of the case the duality between particle and field can ever be completely removed, any more than the distinction between the temporal and the spatial aspects of space-time.

Notes

(1) *Scientific Papers*, (edited by W. D. Niven, 1890, vol. 2, p. 358.
(2) *Ibid.*, pp. 359 ff. See also *A Treatise on Electricity and Magnetism* (1873), Dover edition, 1954, vol. 1, p. ix f, pars. 528–9, pp. 175 ff.
(3) *Scientific Papers*, vol. 2, p. 328. Mathematical symbols need to be correlated with words if they are to fulfil their proper function for it is through word language that their bearing upon reality is interpreted. The same point has been made by Richard Feynman, *The Character of Physical Law*, 1965, p. 55 f.
(4) Cf. Sir James Jeans, *James Clerk Maxwell, A Commemoration Volume 1831–1931*, by J. J. Thomson et al, 1931, p. 102.
(5) Albert Einstein, *ibid.*, p. 56 ff.
(6) *Scientific Papers*, vol. 2, p. 324 f. See also 'On the Proof of the Equations of Motion of a Connected System', *ibid.*, pp. 308–9.
(7) *Scientific Papers*, vol. 2, p. 328.
(8) Cf. the address of H. J. S. Smith announcing Clerk Maxwell's work at the meeting of The British Association, at Bradford in 1873, cited by Sir Oliver Lodge, *Commemorative Volume*, p. 126 f.
(9) *A Treatise on Electricity and Magnetism*, Dover edit., vol. 1, p. viii ff. See also pars. 528–9, pp. 175 ff.
(10) See the essay of 1856, 'Are there Real Analogies in Nature?', reproduced by Lewis Campbell and William Garnett, *The Life of James Clerk Maxwell*, with a Selection from his Correspondence and Occasional Writings and a Sketch of his Contribution to Science, 1882, pp. 235–244. Cf the discussion of Richard Olson, *Scottish Philosophy and British Physics 1750–1880*, Princeton, 1975, pp. 299 ff.
(11) William Berkson, *Fields of Force. The Development of a World View from Faraday to Einstein*, 1974, p. 142.
(12) *A Treatise on Electricity and Magnetism*, vol. 2, par. 529, p. 177.
(13) *Scientific Papers*, vol. 1, pp. 155–229.
(14) *Ibid.*, p. 155 f.
(15) *Ibid.*, p. 160. Cf. p. 187: 'In this outline of Faraday's electrical theories, as they appear from a mathematical point of view, I can do no more than simply state the mathematical methods by which I believe that electrical phenomena can be

best comprehended and reduced to calculation, and my aim has been to present the mathematical ideas to the mind in an *embodied form*, as systems of lines or surfaces, and not as mere symbols, which neither convey the same ideas, nor readily adapt themselves to the phenomena to be explained.' The italics are added.

(16) *Ibid.*, pp. 187 f, 205.
(17) *Ibid.*, p. 208.
(18) *Ibid.*, p. 187.
(19) *Ibid.*, pp. 451–513.
(20) *Ibid.*, p. 452.
(21) Cf. *On Faraday's Lines of Force*, *ibid.*, p. 159.
(22) Sir. J. J. Thomson, *A Commemorative Volume*, p. 53. See Einstein, and Infeld, *The Evolution of Physics*, 1938, p. 140 f.
(23) *Scientific Papers*, vol. 1, p. 468 f. See the helpful explanation of this by W. Berkson, *op. cit.*, pp. 148 ff.
(24) *Scientific Papers*, vol. 1, p. 489 f.
(25) *Ibid.*, p. 486.
(26) *A Treatise on Electricity and Magnetism*, vol. 2, par. 831, p. 470.
(27) William Berkson, *op. cit.*, p. 156.
(28) *Scientific Papers*, vol. 1, p. 500. The italics are Clerk Maxwell's.
(29) Cf. A. Einstein: 'The electromagnetic fields are not states of a medium but independent realities, which cannot be reduced to terms of anything else and are bound to no substratum.' *The World as I See It*, p. 198. See again, Einstein and Infeld, *The Evolution of Physics*, pp. 142 ff. This interpretation is followed by Joseph Agassi, *Faraday as a Natural Philosopher*, 1971, pp. 101 ff, 109 ff, 113, 213 ff, 225 ff, 309 ff; and William Berkson, *op. cit.*, pp. 54 f, 86 f, 172 f, 189.
(30) *Scientific Papers*, vol. 1, pp. 526–597.
(31) *Ibid.*, p.xxi
(32) *Ibid.*, *A Dynamical Theory of the Electromagnetic Field*, par. 3; see below, p. 34.
(33) *Ibid.*, par. 3, p. 34.
(34) *Ibid.*, pars. 6 & 7, p. 35.
(35) *A Treatise on Electricity and Magnetism*, vol. 1, p. viii f; vol. 2, par. 529, p. 176 f. See also the excellent discussion by Richard Olson, *op. cit.*, pp. 299 ff, on Clerk Maxwell's 'embodied mathematics'.
(36) *A Dynamical Theory of the Electromagnetic Field*, pars. 19 & 71, pp. 41, 68 f.
(37) *Ibid.*, par. 20, p. 41 f.
(38) *On Physical Lines of Force*, *Scientific Papers*, vol. 1, pp. 452 ff.
(39) *A Dynamical Theory of the Electromagnetic Field*, pars. 73–4, p. 70.
(40) This is how Einstein interpreted Clerk Maxwell's concept of the field, see below p. 30 f.
(41) See his essay on 'Ether', *Scientific Papers*, vol. 2, pp. 763–775. However, Ivan Tolstoy is probably right when he claims that Clerk Maxwell 'never took models of the aether seriously, preferring to use them as mere analogies—a kind of scaffolding which could later be discarded'. *James Clerk Maxwell. A Biography*, 1981, p. 129. Cf. also J. Agassi, *The Continuing Revolution*, 1968, pp. 218 f.
(42) See here the essay of 1856 on 'Real Analogies in Nature', Campbell and Garnett, *op. cit.*, pp. 237 f, 141 f; and *On Faraday's Lines of Force*, *Scientific Papers*, vol. 1, p. 208.
(43) *A Dynamical Theory of the Electromagnetic Field*, pars. 76 ff, pp. 71 ff.
(44) *Ibid.*, Part VI, pars. 91–108, pp. 82 ff.
(45) *Ibid.*, par. 97, p. 86. Italics added.
(46) Heinrich Hertz, *Electric Waves*, 1893, tr. by D. E. Jones, with a preface by Lord Kelvin, especially pp. 137 ff.

(47) See below, p. 30.
(48) *A Treatise on Electricity and Magnetism*, see especially chs. VI, IX, XX and XXI.
(49) *Ibid.*, vol. 1, p. v f.
(50) *Ibid.*, vol. 1, p. ix f.
(51) *Ibid.*, vol. 2, par. 569, p. 212; par. 574, p. 217; cf. par. 501, p. 157.
(52) *Ibid.*, vol. 2, pars. 568–9, pp. 211 ff.
(53) *Ibid.*, vol. 2, par. 529, p. 176 f.
(54) *An Elementary Treatise on Electricity*, edited by William Garnett, 1881, p. viii. See the note in Campbell and Garnett, *op. cit.*, p. 354.
(55) Cf. Sir J. J Thomson. *A Commemorative Volume*, p. 36 f.
(56) W. Berkson, *op. cit.*, pp. 58 ff, 68 ff, 77, 86 f, 104 f, 116 ff, 124 f. Cf. also J. Agassi, *Faraday as a Natural Philosopher*, 1971, on which Berkson is admittedly dependent, *op. cit.*, p. 343, note 5.
(57) W. Berkson, *op. cit.*, pp. 172 ff, 188.
(58) A Einstein, *The Special and General Theory of Relativity*, 1961 edit., p. 49.
(59) W. Berkson, *op. cit.*, p. 186. Cf. Berkson's note, p. 354, for his reference to Agassi in this respect.
(60) *Op. cit.*, p. 126.
(61) A. Einstein and L. Infeld, *The Evolution of Physics*, pp. 143, 146.
(62) I owe this point to an unpublished lecture delivered by Professor Alan Cook of Cambridge, during the commemoration of the centenary of Clerk Maxwell's death in 1879, 'James Clerk Maxwell. The Man and the Scientist'.
(63) See below, p. 30.

Maxwell's Influence on the Development of the Conception of Physical Reality

ALBERT EINSTEIN

The belief in an external world independent of the observing subject lies at the foundation of all natural science. However, since sense-perceptions only inform us about this external world, or physical reality, indirectly, it is only in a speculative way that it can be grasped by us. Consequently our conceptions of physical reality can never be final. We must always be ready to change these conceptions, i.e. the axiomatic basis of physics, in order to do justice to the facts of observation in the most complete way that is logically possible. In actual fact, a glance at the development of physics shows that this axiomatic basis has met with radical changes from time to time.

The greatest change in the axiomatic basis of physics, and correspondingly in our conception of the structure of reality, since the foundation of theoretical physics through Newton, came about through the researches of Faraday and Maxwell on electromagnetic phenomena. In what follows we shall try to present this in a more precise way, while taking the earlier and later development into account.

In accordance with Newton's system, physical reality is characterised by concepts of space, time, the material point and force (interaction between material points). Physical events are to be thought of as movements according to law of material points in space. The material point is the only representative of reality in so far as it is subject to change. The concept of the material point is obviously due to observable bodies; one conceived of the material point on the analogy of movable bodies by omitting characteristics of extension, form, spatial locality, and all their 'inner' qualities, retaining only inertia, translation, and the additional concept of force. The material bodies which had psychologically given rise to the formation of the concept of 'material point' had now for their part to be conceived as a system of material points. It is to be noted that this theoretical system is essentially atomistic and mechanistic.

All happening was to be conceived of as purely mechanical, that is, merely as motions of material points according to Newton's laws of motion.

The most unsatisfactory aspect of this theoretical system—apart from the difficulty relating to the concept of 'absolute space' which has recently been brought back into the discussion—lay mainly in the doctrine of light, which Newton quite logically had also thought of as consisting of material points. Even at that time the question must already have been felt acutely: What happens to the material points that constitute light, when light itself is absorbed? Moreover, it is altogether unsatisfactory to introduce into the discussion two quite different kinds of material points which had to be put forward to represent ponderable matter and light. Then later on electrical corpuscles were added as a third sort with fundamentally different properties. Besides, it was a weakness in the basic structure that interacting forces had to be postulated quite arbitrarily to account for what happens. Nevertheless, this conception of reality accomplished a lot. How, then, did the conviction arise that it should be abandoned?

In order to give his system mathematical form at all, Newton had first to invent the concept of the differential quotient, and to draw up the laws of motion in the form of total differential equations—perhaps the greatest intellectual step that it has ever been given to one man to take. Partial differential equations were not needed for this, and Newton did not make any methodical use of them. Partial differential equations were needed, however, for the formulation of the mechanics of deformable bodies; this is bound up with the fact that in such problems the way and the manner in which bodies were thought of as constructed out of material points did not play a significant part to begin with.

Thus the partial differential equation came into theoretical physics as a servant, but little by little it took on the role of master. This began in the nineteenth century, when under the pressure of observational facts the undulatory theory of light asserted itself. Light in empty space was conceived as a vibration of the ether, and it seemed idle to conceive of this in turn as a conglomeration of material points. Here for the first time partial differential equations appeared as the natural expression of the primary realities of physics. In a particular area of theoretical physics the continuous field appeared side by side with the material point as the representative of physical reality. This dualism has to this day not disappeared, disturbing as it must be to any systematic mind.

If the idea of physical reality had ceased to be purely atomistic, it still remained purely mechanistic for the time being. One still sought to interpret all happening as the motion of inert bodies: indeed one could not at all imagine any other way of conceiving of things. Then came the great revolution which will be linked with the names of Faraday, Maxwell, Hertz for all time. Maxwell had the lion's share in this revolution. He showed that the whole of what was known at that time about light and electromagnetic phenomena could be represented by his famous double system of partial differential equations, in which the electric and the magnetic fields made their appearance as dependent variables. To be sure Maxwell did try to find a way of grounding or justifying these equations through mechanical thought-models. However, he employed several models of this kind side by side, and took none of them really seriously, so that only the equations themselves appeared as the essential matter, and the field forces which appeared in them as ultimate entities not reducible to anything else. By the turn of the century the conception of the electromagnetic field as an irreducible entity was already generally established and serious theorists had given up confidence in the justification, or the possibility, of a mechanical foundation for Maxwell's equations. Soon, on the contrary, an attempt was made to give a field-theoretical account of material points and their inertia with the help of Maxwell's field theory, but this attempt did not meet with any ultimate success.

If we disregard the important particular results which Maxwell's life work brought about in important areas of physics, and direct attention to the modification which the conception of physical reality experienced through him, we can say: Before Maxwell people thought of physical reality—in so far as it represented events in nature—as material points, whose changes consist only in motions which are subject to total differential equations. After Maxwell they thought of physical reality as represented by continuous fields, not mechanically explicable, which are subject to partial differential equations. This change in the conception of reality is the most profound and the most fruitful that physics has experienced since Newton; but it must also be granted that the complete realisation of the programme implied in this idea has not by any means been carried out yet. The successful systems of physics, which have been set up since then, represent rather compromises between these two programmes, which because of their character as compromises bear the mark of what is provisional and logically incomplete, although in some areas they have made great advances.

Of these the first that must be mentioned is Lorentz's theory of electrons, in which the field and electric corpuscles appear beside one another as equivalent elements in the comprehension of reality. There followed the special and general theory of relativity which—although based entirely on field theory considerations—hitherto could not avoid the independent introduction of material points and total differential equations.

The last and most successful creation of theoretical physics, quantum mechanics, differs fundamentally in its principles from the two programmes which we will briefly designate as Newton's and Maxwell's. For the quantities which appear in its laws lay no claim to describe physical reality itself but only the probabilities for the occurrence of one of the physical realities to which attention is being directed. Dirac, to whom in my judgment, we are indebted for the most logically complete account of this theory, rightly points to the fact that it would not be easy, for example, to give a theoretical description of a photon in such a way that there would be comprised in the description sufficient reason for a judgment as to whether the photon will pass a polarisator set obliquely in its path or not.

Nevertheless, I am inclined to think that physicists will not be satisfied in the long run with this kind of indirect description of reality, even if an adaptation of the theory to the demand of general relativity can be achieved in a satisfactory way. Then they must surely be brought back to the attempt to realise the programme which may suitably be designated as Maxwellian: a description of physical reality in terms of fields which satisfy partial differential equations in a way that is free from singularities.

A Dynamical Theory of the Electromagnetic Field

PART I

INTRODUCTORY

(1) THE most obvious mechanical phenomenon in electrical and magnetical experiments is the mutual action by which bodies in certain states set each other in motion while still at a sensible distance from each other. The first step, therefore, in reducing these phenomena into scientific form, is to ascertain the magnitude and direction of the force acting between the bodies, and when it is found that this force depends in a certain way upon the relative position of the bodies and on their electric or magnetic condition, it seems at first sight natural to explain the facts by assuming the existence of something either at rest or in motion in each body, constituting its electric or magnetic state, and capable of acting at a distance according to mathematical laws.

In this way mathematical theories of statical electricity, of magnetism, of the mechanical action between conductors carrying currents, and of the induction of currents have been formed. In these theories the force acting between the two bodies is treated with reference only to the condition of the bodies and their relative position, and without any express consideration of the surrounding medium.

These theories assume, more or less explicitly, the existence of substances the particles of which have the property of acting on one another at a distance by attraction or repulsion. The most complete development of a theory of this kind is that of M. W. Weber*, who has made the same theory include electrostatic and electromagnetic phenomena.

In doing so, however, he has found it necessary to assume that the force between two electric particles depends on their relative velocity, as well as on their distance.

This theory, as developed by MM. W. Weber and C. Neumann†,

* "Electrodynamische Massbestimmungen". *Leipzig Trans.* Vol. I. 1849, and Taylor's *Scientific Memoirs*, Vol. V. art. XIV.

† *Explicare tentatur quomodo fiat ut lucis planum polarizationis per vires electricas vel magneticas declinetur.*—Halis Saxonum, 1858.

is exceedingly ingenious, and wonderfully comprehensive in its application to the phenomena of statical electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement, and by actually determining electrical quantities with an accuracy hitherto unknown.

(2) The mechanical difficulties, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such as to prevent me from considering this theory as an ultimate one, though it may have been, and may yet be useful in leading to the coordination of phenomena.

I have therefore preferred to seek an explanation of the fact in another direction, by supposing them to be produced by actions which go on in the surrounding medium as well as in the excited bodies, and endeavouring to explain the action between distant bodies without assuming the existence of forces capable of acting directly at sensible distances.

(3) The theory I propose may therefore be called a theory of the *Electromagnetic Field*, because it has to do with the space in the neighbourhood of the electric or magnetic bodies, and it may be called a *Dynamical* Theory, because it assumes that in that space there is matter in motion, by which the observed electromagnetic phenomena are produced.

(4) The electromagnetic field is that part of space which contains and surrounds bodies in electric or magnetic conditions.

It may be filled with any kind of matter, or we may endeavour to render it empty of all gross matter, as in the case of Geissler's tubes and other so-called vacua.

There is always, however, enough of matter left to receive and transmit the undulations of light and heat, and it is because the transmission of these radiations is not greatly altered when transparent bodies of measurable density are substituted for the so-called vacuum, that we are obliged to admit that the undulations are those of an ethereal substance, and not of the gross matter, the presence of which merely modifies in some way the motion of the ether.

We have therefore some reason to believe, from the phenomena of light and heat, that there is an ethereal medium filling space and permeating bodies, capable of being set in motion and of transmitting that motion from one part to another, and of communicating

that motion to gross matter so as to heat it and affect it in various ways.

(5) Now the energy communicated to the body in heating it must have formerly existed in the moving medium, for the undulations had left the source of heat some time before they reached the body, and during that time the energy must have been half in the form of motion of the medium and half in the form of elastic resilience. From these considerations Professor W. Thomson has argued*, that the medium must have a density capable of comparison with that of gross matter, and has even assigned an inferior limit to that density.

(6) We may therefore receive, as a datum derived from a branch of science independent of that with which we have to deal, the existence of a pervading medium, of small but real density, capable of being set in motion, and of transmitting motion from one part to another with great, but not infinite, velocity.

Hence the parts of this medium must be so connected that the motion of one part depends in some way on the motion of the rest; and at the same time these connections must be capable of a certain kind of elastic yielding, since the communication of motion is not instantaneous, but occupies time.

The medium is therefore capable of receiving and storing up two kinds of energy, namely, the "actual" energy depending on the motions of its parts, and "potential" energy, consisting of the work which the medium will do in recovering from displacement in virtue of its elasticity.

The propagation of undulations consists in the continual transformation of one of these forms of energy into the other alternately, and at any instant the amount of energy in the whole medium is equally divided, so that half is energy of motion, and half is elastic resilience.

(7) A medium having such a constitution may be capable of other kinds of motion and displacement than those which produce the phenomena of light and heat, and some of these may be of such a kind that they may be evidenced to our senses by the phenomena they produce.

(8) Now we know that the luminiferous medium is in certain cases acted on by magnetism; for Faraday† discovered that when a

* "On the Possible Density of the Luminiferous Medium, and on the Mechanical Value of a Cubic Mile of Sunlight", *Transactions of the Royal Society of Edinburgh* (1854), p. 57.

† *Experimental Researches*, Series XIX.

plane polarized ray traverses a transparent diamagnetic medium in the direction of the lines of magnetic force produced by magnets or currents in the neighbourhood, the plane of polarization is caused to rotate.

This rotation is always in the direction in which positive electricity must be carried round the diamagnetic body in order to produce the actual magnetization of the field.

M. Verdet* has since discovered that if a paramagnetic body, such as solution of perchloride of iron in ether, be substituted for the diamagnetic body, the rotation is in the opposite direction.

Now Professor W. Thomson† has pointed out that no distribution of forces acting between the parts of a medium whose only motion is that of the luminous vibrations, is sufficient to account for the phenomena, but that we must admit the existence of a motion in the medium depending on the magnetization, in addition to the vibratory motion which constitutes light.

It is true that the rotation by magnetism of the plane of polarization has been observed only in media of considerable density; but the properties of the magnetic field are not so much altered by the substitution of one medium for another, or for a vacuum, as to allow us to suppose that the dense medium does anything more than merely modify the motion of the ether. We have therefore warrantable grounds for inquiring whether there may not be a motion of the ethereal medium going on wherever magnetic effects are observed, and we have some reason to suppose that this motion is one of rotation, having the direction of the magnetic force as its axis.

(9) We may now consider another phenomenon observed in the electromagnetic field. When a body is moved across the lines of magnetic force it experiences what is called an electromotive force; the two extremities of the body tend to become oppositely electrified, and an electric current tends to flow through the body. When the electromotive force is sufficiently powerful, and is made to act on certain compound bodies, it decomposes them, and causes one of their components to pass towards one extremity of the body, and the other in the opposite direction.

Here we have evidence of a force causing an electric current in spite of resistance; electrifying the extremities of a body in opposite ways, a condition which is sustained only by the action of the

* *Comptes Rendus* (1856, second half year, p. 529, and 1857, first half year, p. 1209).

† *Proceedings of the Royal Society*, June 1856 and June 1861.

electromotive force, and which, as soon as that force is removed, tends, with an equal and opposite force, to produce a counter current through the body and to restore the original electrical state of the body; and finally, if strong enough, tearing to pieces chemical compounds and carrying their components in opposite directions, while their natural tendency is to combine, and to combine with a force which can generate an electromotive force in the reverse direction.

This, then, is a force acting on a body caused by its motion through the electromagnetic field, or by changes occurring in that field itself; and the effect of the force is either to produce a current and heat the body, or to decompose the body, or, when it can do neither, to put the body in a state of electric polarization—a state of constraint in which opposite extremities are oppositely electrified, and from which the body tends to relieve itself as soon as the disturbing force is removed.

(10) According to the theory which I propose to explain, this "electromotive force" is the force called into play during the communication of motion from one part of the medium to another, and it is by means of this force that the motion of one part causes motion in another part. When electromotive force acts on a conducting circuit, it produces a current, which, as it meets with resistance, occasions a continual transformation of electrical energy into heat, which is incapable of being restored again to the form of electrical energy by any reversal of the process.

(11) But when electromotive force acts on a dielectric it produces a state of polarization of its parts similar in distribution to the polarity of the parts of a mass of iron under the influence of a magnet, and like the magnetic polarization, capable of being described as a state in which every particle has its opposite poles in opposite conditions*.

In a dielectric under the action of electromotive force, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another. The effect of this action on the whole dielectric mass is to produce a general displacement of electricity in a certain direction. This displacement does not

* Faraday, *Experimental Researches*, Series XI.; Mossotti, *Mem. della Soc. Italiana* (Modena), Vol. XXIV. Part 2, p. 49.

amount to a current, because when it has attained to a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or the negative direction according as the displacement is increasing or decreasing. In the interior of the dielectric there is no indication of electrification, because the electrification of the surface of any molecule is neutralized by the opposite electrification of the surface of the molecules in contact with it; but at the bounding surface of the dielectric, where the electrification is not neutralized, we find the phenomena which indicate positive or negative electrification.

The relation between the electromotive force and the amount of electric displacement it produces depends on the nature of the dielectric, the same electromotive force producing generally a greater electric displacement in solid dielectrics, such as glass or sulphur, than in air.

(12) Here, then, we perceive another effect of electromotive force, namely, electric displacement, which according to our theory is a kind of elastic yielding to the action of the force, similar to that which takes place in structures and machines owing to the want of perfect rigidity of the connexions.

(13) The practical investigation of the inductive capacity of dielectrics is rendered difficult on account of two disturbing phenomena. The first is the conductivity of the dielectric, which, though in many cases exceedingly small, is not altogether insensible. The second is the phenomenon called electric absorption*, in virtue of which, when the dielectric is exposed to electromotive force, the electric displacement gradually increases, and when the electromotive force is removed, the dielectric does not instantly return to its primitive state, but only discharges a portion of its electrification, and when left to itself gradually acquires electrification on its surface, as the interior gradually becomes depolarized. Almost all solid dielectrics exhibit this phenomenon, which gives rise to the residual charge in the Leyden jar, and to several phenomena of electric cables described by Mr F. Jenkin†.

(14) We have here two other kinds of yielding besides the yielding of the perfect dielectric, which we have compared to a perfectly elastic body. The yielding due to conductivity may be compared to

* Faraday, *Experimental Researches*, 1233–1250.

† *Reports of British Association*, 1859, p. 248; and *Report of Committee of Board of Trade on Submarine Cables*, pp. 136 & 464.

that of a viscous fluid (that is to say, a fluid having great internal friction), or a soft solid on which the smallest force produces a permanent alteration of figure increasing with the time during which the force acts. The yielding due to electric absorption may be compared to that of a cellular elastic body containing a thick fluid in its cavities. Such a body, when subjected to pressure, is compressed by degrees on account of the gradual yielding of the thick fluid; and when the pressure is removed it does not at once recover its figure, because the elasticity of the substance of the body has gradually to overcome the tenacity of the fluid before it can regain complete equilibrium.

Several solid bodies in which no such structure as we have supposed can be found, seem to possess a mechanical property of this kind*; and it seems probable that the same substances, if dielectrics, may possess the analogous electrical property, and if magnetic, may have corresponding properties relating to the acquisition, retention, and loss of magnetic polarity.

(15) It appears therefore that certain phenomena in electricity and magnetism lead to the same conclusion as those of optics, namely, that there is an ethereal medium pervading all bodies, and modified only in degree by their presence; that the parts of this medium are capable of being set in motion by electric currents and magnets; that this motion is communicated from one part of the medium to another by forces arising from the connections of those parts; that under the action of these forces there is a certain yielding depending on the elasticity of these connections; and that therefore energy in two different forms may exist in the medium, the one form being the actual energy of motion of its parts, and the other being the potential energy stored up in the connections, in virtue of their elasticity.

(16) Thus, then, we are led to the conception of a complicated mechanism capable of a vast variety of motion, but at the same time so connected that the motion of one part depends, according to definite relations, on the motion of other parts, these motions being communicated by forces arising from the relative displacement of the connected parts, in virtue of their elasticity. Such a mechanism must be subject to the general laws of Dynamics, and we ought to be able to work out all the consequences of its motion, provided we

* As, for instance, the composition of glue, treacle, &c., of which small plastic figures are made, which after being distorted gradually recover their shape.

know the form of the relation between the motions of the parts.

(17) We know that when an electric current is established in a conducting circuit, the neighbouring part of the field is characterized by certain magnetic properties, and that if two circuits are in the field, the magnetic properties of the field due to the two currents are combined. Thus each part of the field is in connection with both currents, and the two currents are put in connection with each other in virtue of their connection with the magnetization of the field. The first result of this connection that I propose to examine, is the induction of one current by another, and by the motion of conductors in the field.

The second result, which is deduced from this, is the mechanical action between conductors carrying currents. The phenomenon of the induction of currents has been deduced from their mechanical action by Helmholtz* and Thomson†. I have followed the reverse order, and deduced the mechanical action from the laws of induction. I have then described experimental methods of determining the quantities L, M, N, on which these phenomena depend.

(18) I then apply the phenomena of induction and attraction of currents to the exploration of the electromagnetic field, and the laying down systems of lines of magnetic force which indicate its magnetic properties. By exploring the same field with a magnet, I show the distribution of its equipotential magnetic surfaces, cutting the lines of force at right angles.

In order to bring these results within the power of symbolical calculation, I then express them, in the form of the General Equations of the Electromagnetic Field. These equations express:

(A) The relation between electric displacement, true conduction, and the total current, compounded of both.

(B) The relation between the lines of magnetic force and the inductive coefficients of a circuit, as already deduced from the laws of induction.

(C) The relation between the strength of a current and its magnetic effects, according to the electromagnetic system of measurement.

* "Conservation of Force", *Physical Society of Berlin*, 1847; and Taylor's *Scientific Memoirs*, 1853, p. 114.

† *Reports of the British Association*, 1848; *Philosophical Magazine*, Dec. 1851.

(D) The value of the electromotive force in a body, as arising from the motion of the body in the field, the alteration of the field itself, and the variation of electric potential from one part of the field to another.

(E) The relation between electric displacement, and the electromotive force which produces it.

(F) The relation between an electric current, and the electromotive force which produces it.

(G) The relation between the amount of free electricity at any point, and the electric displacements in the neighbourhood.

(H) The relation between the increase or diminution of free electricity and the electric currents in the neighbourhood.

There are twenty of these equations in all, involving twenty variable quantities.

(19) I then express in terms of these quantities the intrinsic energy of the Electromagnetic Field as depending partly on its magnetic and partly on its electric polarization at every point.

From this I determine the mechanical force acting, 1st, on a moveable conductor carrying an electric current; 2ndly, on a magnetic pole; 3rdly, on an electrified body.

The last result, namely, the mechanical force acting on an electrified body, gives rise to an independent method of electrical measurement founded on its electrostatic effects. The relation between the units employed in the two methods is shown to depend on what I have called the "electric elasticity" of the medium, and to be a velocity, which has been experimentally determined by MM. Weber and Kohlrausch.

I then show how to calculate the electrostatic capacity of a condenser, and the specific inductive capacity of a dielectric.

The case of a condenser composed of parallel layers of substances of different electric resistances and inductive capacities is next examined, and it is shown that the phenomenon called electric absorption will generally occur, that is, the condenser, when suddenly discharged, will after a short time show signs of a *residual* charge.

(20) The general equations are next applied to the case of a magnetic disturbance propagated through a non-conducting field, and it is shown that the only disturbances which can be so propagated are those which are transverse to the direction of propagation,

and that the velocity of propagation is the velocity v, found from experiments such as those of Weber, which expresses the number of electrostatic units of electricity which are contained in one electromagnetic unit.

This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws. If so, the agreement between the elasticity of the medium as calculated from the rapid alternations of luminous vibrations, and as found by the slow processes of electrical experiments, shows how perfect and regular the elastic properties of the medium must be when not encumbered with any matter denser than air. If the same character of the elasticity is retained in dense transparent bodies, it appears that the square of the index of refraction is equal to the product of the specific dielectric capacity and the specific magnetic capacity. Conducting media are shown to absorb such radiations rapidly, and therefore to be generally opaque.

The conception of the propagation of transverse magnetic disturbances to the exclusion of normal ones is distinctly set forth by Professor Faraday* in his "Thoughts on Ray Vibrations". The electromagnetic theory of light, as proposed by him, is the same in substance as that which I have begun to develop in this paper, except that in 1846 there were no data to calculate the velocity of propagation.

(21) The general equations are then applied to the calculation of the coefficients of mutual induction of two circular currents and the coefficient of self-induction in a coil. The want of uniformity of the current in the different parts of the section of a wire at the commencement of the current is investigated, I believe for the first time, and the consequent correction of the coefficient of self-induction is found.

These results are applied to the calculation of the self-induction of the coil used in the experiments of the Committee of the British Association on Standards of Electric Resistance, and the value compared with that deduced from the experiments.

* *Philosophical Magazine*, May 1846, or *Experimental Researches*, III. p. 447.

PART II

ON ELECTROMAGNETIC INDUCTION

Electromagnetic Momentum of a Current

(22) We may begin by considering the state of the field in the neighbourhood of an electric current. We know that magnetic forces are excited in the field, their direction and magnitude depending according to known laws upon the form of the conductor carrying the current. When the strength of the current is increased, all the magnetic effects are increased in the same proportion. Now, if the magnetic state of the field depends on motions of the medium, a certain force must be exerted in order to increase or diminish these motions, and when the motions are excited they continue, so that the effect of the connection between the current and the electromagnetic field surrounding it, is to endow the current with a kind of momentum, just as the connection between the driving-point of a machine and a fly-wheel endows the driving-point with an additional momentum, which may be called the momentum of the fly-wheel reduced to the driving-point. The unbalanced force acting on the driving-point increases this momentum, and is measured by the rate of its increase.

In the case of electric currents, the resistance to sudden increase or diminution of strength produces effects exactly like those of momentum, but the amount of this momentum depends on the shape of the conductor and the relative position of its different parts.

Mutual Action of Two Currents

(23) If there are two electric currents in the field, the magnetic force at any point is that compounded of the forces due to each current separately, and since the two currents are in connection with every point of the field, they will be in connection with each other, so that any increase or diminution of the one will produce a force acting with or contrary to the other.

Dynamical Illustration of Reduced Momentum

(24) As a dynamical illustration, let us suppose a body C so connected with two independent driving-points A and B that its velocity is p times that of A together with q times that of B. Let u

be the velocity of A, v that of B, and w that of C, and let δx, δy, δz be their simultaneous displacements, then by the general equation of dynamics*,

$$C\frac{dw}{dt}\,\delta z = X\,\delta x + Y\,\delta y,$$

where X and Y are the forces acting at A and B.

But
$$\frac{dw}{dt} = p\frac{du}{dt} + q\frac{dv}{dt},$$

and
$$\delta z = p\delta x + q\delta y.$$

Substituting, and remembering that δx and δy are independent,

$$\left.\begin{aligned} X &= \frac{d}{dt}(Cp^2u + Cpqv) \\ Y &= \frac{d}{dt}(Cpqu + Cq^2v) \end{aligned}\right\} \qquad (1)$$

We may call $Cp^2u + Cpqv$ the momentum of C referred to A, and $Cpqu + Cq^2v$ its momentum referred to B; then we may say that the effect of the force X is to increase the momentum of C referred to A, and that of Y to increase its momentum referred to B.

If there are many bodies connected with A and B in a similar way but with different values of p and q, we may treat the question in the same way by assuming

$$L = \sum(Cp^2), \qquad M = \sum(Cpq), \quad \text{and} \quad N = \sum(Cq^2),$$

where the summation is extended to all the bodies with their proper values of C, p, and q. Then the momentum of the system referred to A is

$$Lu + Mv,$$

and referred to B,
$$Mu + Nv,$$

and we shall have
$$\left.\begin{aligned} X &= \frac{d}{dt}(Lu + Mv) \\ Y &= \frac{d}{dt}(Mu + Nv) \end{aligned}\right\}, \qquad (2)$$

where X and Y are the external forces acting on A and B.

* Lagrange, *Méc. Anal.* II. 2, § 5.

(25) To make the illustration more complete we have only to suppose that the motion of A is resisted by a force proportional to its velocity, which we may call Ru, and that of B by a similar force, which we may call Sv, R and S being coefficients of resistance. Then if ξ and η are the forces on A and B,

$$\left.\begin{aligned} \xi &= X+Ru = Ru+\frac{d}{dt}(Lu+Mv) \\ \eta &= Y+Sv = Sv+\frac{d}{dt}(Mu+Nv) \end{aligned}\right\} \tag{3}$$

If the velocity of A be increased at the rate $\dfrac{du}{dt}$, then in order to prevent B from moving a force, $\eta=\dfrac{d}{dt}(Mu)$ must be applied to it.

This effect on B, due to an increase of the velocity of A, corresponds to the electromotive force on one circuit arising from an increase in the strength of a neighbouring circuit.

This dynamical illustration is to be considered merely as assisting the reader to understand what is meant in mechanics by Reduced Momentum. The facts of the induction of currents as depending on the variations of the quantity called Electromagnetic Momentum, or Electrotonic State, rest on the experiments of Faraday*, Felici†, &c.

Coefficients of Induction for Two Circuits

(26) In the electromagnetic field the values of L, M, N depend on the distribution of the magnetic effects due to the two circuits, and this distribution depends only on the form and relative position of the circuits. Hence L, M, N are quantities depending on the form and relative position of the circuits, and are subject to variation with the motion of the conductors. It will be presently seen that L, M, N are geometrical quantities of the nature of lines that is, of one dimension in space; L depends on the form of the first conductor, which we shall call A, N on that of the second, which we shall call B, and M on the relative position of A and B.

(27) Let ξ be the electromotive force acting on A, x the strength of the current, and R the resistance, then Rx will be the resisting force. In steady currents the electromotive force just balances the

* *Experimental Researches*, Series I., IX.

† *Annales de Chimie*, sér. 3, XXXIV. (1852), p. 64.

resisting force, but in variable currents the resultant force $\xi - Rx$ is expended in increasing the "electromagnetic momentum", using the word momentum merely to express that which is generated by a force acting during a time, that is, a velocity existing in a body.

In the case of electric currents, the force in action is not ordinary mechanical force, at least we are not as yet able to measure it as common force, but we call it electromotive force, and the body moved is not merely the electricity in the conductor, but something outside the conductor, and capable of being affected by other conductors in the neighbourhood carrying currents. In this it resembles rather the reduced momentum of a driving-point of a machine as influenced by its mechanical connections, than that of a simple moving body like a cannon ball, or water in a tube.

Electromagnetic Relations of Two Conducting Circuits

(28) In the case of two conducting circuits, A and B, we shall assume that the electromagnetic momentum belonging to A is

$$Lx + My,$$

and that belonging to B, $\quad Mx + Ny,$

where L, M, N correspond to the same quantities in the dynamical illustration, except that they are supposed to be capable of variation when the conductors A or B are moved.

Then the equation of the current x in A will be

$$\xi = Rx + \frac{d}{dt}(Lx + My), \tag{4}$$

and that of y in B

$$\eta = Sy + \frac{d}{dt}(Mx + Ny), \tag{5}$$

where ξ and η are the electromotive forces, x and y the currents, and R and S the resistances in A and B respectively.

Induction of One Current by Another

(29) Case 1st. Let there be no electromotive force on B, except that which arises from the action of A, and let the current of A increase from 0 to the value x, then

$$Sy + \frac{d}{dt}(Mx + Ny) = 0,$$

whence $$Y = \int_0^t y\,dt = -\frac{M}{S}x, \tag{6}$$

that is, a quantity of electricity Y, being the total induced current, will flow through B when x rises from 0 to x. This is induction by variation of the current in the primary conductor. When M is positive, the induced current due to increase of the primary current is negative.

Induction by Motion of Conductor

(30) Case 2nd. Let x remain constant, and let M change from M to M', then

$$Y = -\frac{M' - M}{S}x; \tag{7}$$

so that if M is increased, which it will be by the primary and secondary circuits approaching each other, there will be a negative induced current, the total quantity of electricity passed through B being Y.

This is induction by the relative motion of the primary and secondary conductors.

Equation of Work and Energy.

(31) To form the equation between work done and energy produced, multiply (1) by x and (2) by y, and add

$$\xi x + \eta y = Rx^2 + Sy^2 + x\frac{d}{dt}(Lx + My) + y\frac{d}{dt}(Mx + Ny). \tag{8}$$

Here ξx is the work done in unit of time by the electromotive force ξ acting on the current x and maintaining it, and ηy is the work done by the electromotive force η. Hence the left-hand side of the equation represents the work done by the electromotive forces in unit of time.

Heat Produced by the Current

(32) On the other side of the equation we have, first,

$$Rx^2 + Sy^2 = H, \tag{9}$$

which represents the work done in overcoming the resistance of the circuits in unit of time. This is converted into heat. The remaining

terms represent work not converted into heat. They may be written

$$\tfrac{1}{2}\frac{d}{dt}(Lx^2+2Mxy+Ny^2)+\tfrac{1}{2}\frac{dL}{dt}x^2+\frac{dM}{dt}xy+\tfrac{1}{2}\frac{dN}{dt}y^2.$$

Intrinsic Energy of the Currents

(33) If L, M, N are constant, the whole work of the electromotive forces which is not spent against resistance will be devoted to the development of the currents. The whole intrinsic energy of the currents is therefore

$$\tfrac{1}{2}Lx^2+Mxy+\tfrac{1}{2}Ny^2=E. \tag{10}$$

This energy exists in a form imperceptible to our senses, probably as actual motion, the seat of this motion being not merely the conducting circuits, but the space surrounding them.

Mechanical Action Between Conductors

(34) The remaining terms,

$$\tfrac{1}{2}\frac{dL}{dt}x^2+\frac{dM}{dt}xy+\tfrac{1}{2}\frac{dN}{dt}y^2=W, \tag{11}$$

represent the work done in unit of time arising from the variations of L, M, and N, or, what is the same thing, alterations in the form and position of the conducting circuits A and B.

Now if work is done when a body is moved, it must arise from ordinary mechanical force acting on the body while it is moved. Hence this part of the expression shows that there is a mechanical force urging every part of the conductors themselves in that direction in which L, M, and N will be most increased.

The existence of the electromagnetic force between conductors carrying currents is therefore a direct consequence of the joint and independent action of each current on the electromagnetic field. If A and B are allowed to approach a distance ds, so as to increase M from M to M' while the currents are x and y, then the work done will be

$$(M'-M)xy,$$

and the force in the direction of ds will be

$$\frac{dM}{ds}xy, \tag{12}$$

and this will be an attraction if x and y are of the same sign, and if M is increased as A and B approach.

It appears, therefore, that if we admit that the unresisted part of electromotive force goes on as long as it acts, generating a self-persistent state of the current, which we may call (from mechanical analogy) its electromagnetic momentum, and that this momentum depends on circumstances external to the conductor, then both induction of currents and electromagnetic attractions may be proved by mechanical reasoning.

What I have called electromagnetic momentum is the same quantity which is called by Faraday* the electrotonic state of the circuit, every change of which involves the action of an electromotive force, just as change of momentum involves the action of mechanical force.

If, therefore, the phenomena described by Faraday in the Ninth Series of his *Experimental Researches* were the only known facts about electric currents, the laws of Ampère relating to the attraction of conductors carrying currents, as well as those of Faraday about the mutual induction of currents, might be deduced by mechanical reasoning.

In order to bring these results within the range of experimental verification, I shall next investigate the case of a single current, of two currents, and of the six currents in the electric balance, so as to enable the experimenter to determine the values of L, M, N.

Case of a Single Circuit

(35) The equation of the current x in a circuit whose resistance is R, and whose coefficient of self-induction is L, acted on by an external electromotive force ξ, is

$$\xi - Rx = \frac{d}{dt} Lx. \tag{13}$$

When ξ is constant, the solution is of the form

$$x = b + (a-b)e - \frac{R}{L} t,$$

where a is the value of the current at the commencement, and b is its final value.

* *Experimental Researches*, Series I. 60, &c.

The total quantity of electricity which passes in time t, where t is great, is

$$\int_0^t x\,dt = bt + (a-b)\frac{L}{R}. \tag{14}$$

The value of the integral of x^2 with respect to the time is

$$\int_0^t x^2\,dt = b^2 t + (a-b)\frac{L}{R}\left(\frac{3b+a}{2}\right). \tag{15}$$

The actual current changes gradually from the initial value a to the final value b, but the values of the integrals of x and x^2 are the same as if a steady current of intensity $\frac{1}{2}(a+b)$ were to flow for a time $2\frac{L}{R}$, and were then succeeded by the steady current b. The time $2\frac{L}{R}$ is generally so minute a fraction of a second, that the effects on the galvanometer and dynamometer may be calculated as if the impulse were instantaneous.

If the circuit consists of a battery and a coil, then, when the circuit is first completed, the effects are the same as if the current had only half its final strength during the time $2\frac{L}{R}$. This diminution of the current, due to induction, is sometimes called the counter-current.

(36) If an additional resistance r is suddenly thrown into the circuit, as by breaking contact, so as to force the current to pass through a thin wire of resistance r, then the original current is $a = \frac{\xi}{R}$, and the final current is $b = \frac{\xi}{R+r}$.

The current of induction is then $\frac{1}{2}\xi\frac{2R+r}{R(R+r)}$, and continues for a time $2\frac{L}{R+r}$. This current is greater than that which the battery can maintain in the two wires R and r, and may be sufficient to ignite the thin wire r.

When contact is broken by separating the wires in air, this additional resistance is given by the interposed air, and since the electromotive force across the new resistance is very great, a spark will be forced across.

If the electromotive force is of the form $E \sin pt$, as in the case of

a coil revolving in the magnetic field, then

$$x = \frac{E}{\rho} \sin(pt - \alpha),$$

where $\rho^2 = R^2 + L^2p^2$, and $\tan \alpha = \dfrac{Lp}{R}$.

Case of Two Circuits

(37) Let R be the primary circuit and S the secondary circuit, then we have a case similar to that of the induction coil.

The equations of currents are those marked A and B, and we may here assume L, M, N as constant because there is no motion of the conductors. The equations then become

$$\left.\begin{aligned} Rx + L\frac{dx}{dt} + M\frac{dy}{dt} &= \xi \\ Sy + M\frac{dx}{dt} + N\frac{dy}{dt} &= 0 \end{aligned}\right\}. \tag{13*}$$

To find the total quantity of electricity which passes, we have only to integrate these equations with respect to t; then if x_0, y_0 be the strengths of the currents at time 0, and x_1, y_1 at time t, and if X, Y be the quantities of electricity passed through each circuit during time t,

$$\left.\begin{aligned} X &= \frac{1}{R}\{\xi t + L(x_0 - x_1) + M(y_0 - y_1)\} \\ Y &= \frac{1}{S}\{M(x_0 - x_1) + N(y_0 - y_1)\} \end{aligned}\right\} \tag{14*}$$

When the circuit R is completed, then the total currents up to time t, when t is great, are found by making

$$x_0 = 0, \qquad x_1 = \frac{\xi}{R}, \qquad y_0 = 0, \qquad y_1 = 0;$$

then

$$X = x_1\left(t - \frac{L}{R}\right), \qquad Y = -\frac{M}{S}x_1. \tag{15*}$$

The value of the total counter-current in R is therefore independent of the secondary circuit, and the induction current in the secondary circuit depends only on M, the coefficient of induction

between the coils, S the resistance of the secondary coil, and x_1 the final strength of the current in R.

When the electromotive force ξ ceases to act, there is an extra current in the primary circuit, and a positive induced current in the secondary circuit, whose values are equal and opposite to those produced on making contact.

(38) All questions relating to the total quantity of transient currents, as measured by the impulse given to the magnet of the galvanometer, may be solved in this way without the necessity of a complete solution of the equations. The heating effect of the current, and the impulse it gives to the suspended coil of Weber's dynamometer, depend on the square of the current at every instant during the short time it lasts. Hence we must obtain the solution of the equations, and from the solution we may find the effects both on the galvanometer and dynamometer; and we may then make use of the method of Weber for estimating the intensity and duration of a current uniform while it lasts which would produce the same effects.

(39) Let n_1, n_2 be the roots of the equation

$$(LN-M^2)n^2+(RN+LS)n+RS=0, \tag{16}$$

and let the primary coil be acted on by a constant electromotive force Rc, so that c is the constant current it could maintain; then the complete solution of the equations for making contact is

$$x=\frac{c}{S}\frac{n_1n_2}{n_1-n_2}\left\{\left(\frac{S}{n_1}+N\right)e^{n_1t}-\left(\frac{S}{n_2}+N\right)e^{n_2t}+S\frac{n_1-n_2}{n_1n_2}\right\}, \tag{17}$$

$$y=\frac{cM}{S}\frac{n_1n_2}{n_1-n_2}\{e^{n_1t}-e^{n_2t}\} \tag{18}$$

From these we obtain for calculating the impulse on the dynamometer,

$$\int x^2dt=c^2\left\{t-\tfrac{3}{2}\frac{L}{R}-\tfrac{1}{2}\frac{M^2}{RN+LS}\right\}, \tag{19}$$

$$\int y^2dt=c^2\tfrac{1}{2}\frac{M^2R}{S(RN+LS)}. \tag{20}$$

The effects of the current in the secondary coil on the galvanometer and dynamometer are the same as those of a uniform current

$$-\tfrac{1}{2}c\frac{MR}{RN+LS}$$

for a time $$2\left(\frac{L}{R}+\frac{N}{S}\right).$$

(40) The equation between work and energy may be easily verified. The work done by the electromotive force is

$$\xi\int x dt = c^2(Rt-L).$$

Work done in overcoming resistance and producing heat,

$$R\int x^2 dt + S\int y^2 dt = c^2(Rt-\tfrac{3}{2}L).$$

Energy remaining in the system, $=\tfrac{1}{2}c^2L.$

(41) If the circuit R is suddenly and completely interrupted while carrying a current c, then the equation of the current in the secondary coil would be

$$y = c\frac{M}{N}e-\frac{S}{N}t.$$

This current begins with a value $c\dfrac{M}{N}$, and gradually disappears.

The total quantity of electricity is $c\dfrac{M}{S}$, and the value of $\int y^2 dt$ is $c^2\dfrac{M^2}{2SN}$.

The effects on the galvanometer and dynamometer are equal to those of a uniform current $\tfrac{1}{2}c\dfrac{M}{N}$ for a time $2\dfrac{N}{S}$.

The heating effect is therefore greater than that of the current on making contact.

(42) If an electromotive force of the form $\xi = E\cos pt$ acts on the circuit R, then if the circuit S is removed, the value of x will be

$$x=\frac{E}{A}\sin(pt-\alpha),$$

where $$A^2=R^2+L^2p^2,$$

and $$\tan\alpha=\frac{Lp}{R}.$$

The effect of the presence of the circuit S in the neighbourhood is to alter the value of A and α, to that which they would be if R

became

$$R+p^2\frac{MS}{S^2+p^2N^2},$$

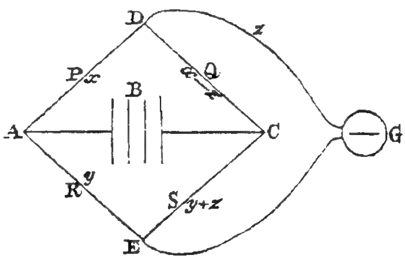
and L became

$$L-p^2\frac{MN}{S^2+p^2N^2}.$$

Hence the effect of the presence of the circuit S is to increase the apparent resistance and diminish the apparent self-induction of the circuit R.

On the Determination of Coefficients of Induction by the Electric Balance.

(43) The electric balance consists of six conductors joining four points, A, C, D, E, two and two. One pair, AC, of these points is connected through the battery B. The opposite pair, DE, is connected through the galvanometer G. Then if the resistances of the



four remaining conductors are represented by P, Q, R, S, and the currents in them by x, $x-z$, y, and $y+z$, the current through G will be z. Let the potentials at the four points be A, C, D, E. Then the conditions of steady currents may be found from the equations

$$\left.\begin{array}{ll} Px=A-D, & Q(x-z)=D-C \\ Ry=A-E, & S(y+z)=E-C \\ Gz=D-E, & B(x+y)=-A+C+F \end{array}\right\}. \tag{21}$$

Solving these equations for z, we find

$$z\left\{\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+B\left(\frac{1}{P}+\frac{1}{R}\right)\left(\frac{1}{Q}+\frac{1}{S}\right)+G\left(\frac{1}{P}+\frac{1}{Q}\right)\left(\frac{1}{R}+\frac{1}{S}\right)\right.$$
$$\left.+\frac{BG}{PQRS}(P+Q+R+S)\right\}=F\left(\frac{1}{PS}-\frac{1}{QR}\right). \tag{22}$$

In this expression F is the electromotive force of the battery, z the current through the galvanometer when it has become steady. P,

Q, R, S the resistances in the four arms. B that of the battery and electrodes, and G that of the galvanometer.

(44) If $PS = QR$, then $z = 0$, and there will be no steady current, but a transient current through the galvanometer may be produced on making or breaking circuit on account of induction, and the indications of the galvanometer may be used to determine the coefficients of induction, provided we understand the actions which take place.

We shall suppose $PS = QR$, so that the current z vanishes when sufficient time is allowed, and

$$x(P+Q) = y(R+S) = \frac{F(P+Q)(R+S)}{(P+Q)(R+S)+B(P+Q)(R+S)}. \tag{23}$$

Let the induction coefficients between P, Q, R, S be given by the following table, the coefficient of induction of P on itself being p, between P and Q, h, and so on.

	P	Q	R	S
P	p	h	k	l
Q	h	q	m	n
R	k	m	r	o
S	l	n	o	s

Let g be the coefficient of induction of the galvanometer on itself, and let it be out of the reach of the inductive influence of P, Q, R, S (as it must be in order to avoid direct action of P, Q, R, S on the needle). Let X, Y, Z be the integrals of x, y, z with respect to t. At making contact x, y, z are zero. After a time z disappears, and x and y reach constant values. The equations for each conductor will therefore be

$$\left.\begin{aligned}
PX+(p+h)x+(k+l)y &= \int A\,dt - \int D\,dt \\
Q(X-Z)+(h+q)x+(m+n)y &= \int D\,dt - \int C\,dt \\
RY+(k+m)x+(r+o)y &= \int A\,dt - \int E\,dt \\
S(Y+Z)+(l+n)x+(o+s)y &= \int E\,dt - \int C\,dt \\
GZ &= \int D\,dt - \int E\,dt.
\end{aligned}\right\} \tag{24}$$

Solving these equations for Z, we find

$$\left.\begin{aligned} Z\Big\{\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}+\frac{1}{S}+B\left(\frac{1}{P}+\frac{1}{R}\right)\left(\frac{1}{Q}+\frac{1}{S}\right)+G\left(\frac{1}{P}+\frac{1}{Q}\right)\left(\frac{1}{R}+\frac{1}{S}\right)& \\ +\frac{BG}{PQRS}(P+Q+R+S)\Big\}& \\ =-F\frac{1}{PS}\Big\{\frac{p}{P}-\frac{q}{Q}-\frac{r}{R}+\frac{s}{S}+h\left(\frac{1}{P}-\frac{1}{Q}\right)+k\left(\frac{1}{R}-\frac{1}{P}\right)+l\left(\frac{1}{R}+\frac{1}{Q}\right)& \\ -m\left(\frac{1}{P}+\frac{1}{S}\right)+n\left(\frac{1}{Q}-\frac{1}{S}\right)+o\left(\frac{1}{S}-\frac{1}{R}\right)\Big\}& \end{aligned}\right\}. \qquad (25)$$

(45) Now let the deflection of the galvanometer by the instantaneous current whose intensity is Z be α.

* Let the permanent deflection produced by making the ratio of PS to QR, ρ instead of unity, be θ.

Also let the time of vibration of the galvanometer needle from rest to rest be T.

Then calling the quantity

$$\frac{p}{P}-\frac{q}{Q}-\frac{r}{R}+\frac{s}{S}+h\left(\frac{1}{P}-\frac{1}{Q}\right)+k\left(\frac{1}{R}-\frac{1}{P}\right)+l\left(\frac{1}{R}+\frac{1}{Q}\right)-m\left(\frac{1}{P}+\frac{1}{S}\right)$$
$$+n\left(\frac{1}{Q}-\frac{1}{S}\right)+o\left(\frac{1}{S}-\frac{1}{R}\right)=\tau, \qquad (26)$$

we find

$$\frac{Z}{z}=\frac{2\sin\frac{1}{2}\alpha}{\tan\theta}\frac{T}{\pi}=\frac{\tau}{1-\rho}. \qquad (27)$$

In determining τ by experiment, it is best to make the alteration of resistance in one of the arms by means of the arrangement described by Mr Jenkin in the Report of the British Association for 1863, by which any value of ρ from 1 to 1·01 can be accurately measured.

We observe (α) the greatest deflection due to the impulse of induction when the galvanometer is in circuit, when the connections are made, and when the resistances are so adjusted as to give no permanent current.

* [In these circumstances the values of x and y found in Art. 44 require modification before being inserted in equation (24). This has been pointed out by Lord Rayleigh, who employed the method described in the text in his second determination of the British unit of resistance in absolute measure. See the *Philosophical Transactions*, 1882. Part II. pp. 677, 678.]

We then observe (β) the greatest deflection produced by the permanent current when the resistance of one of the arms is increased in the ratio of 1 to ρ, the galvanometer not being in circuit till a little while after the connection is made with the battery.

In order to eliminate the effects of resistance of the air, it is best to vary ρ till $\beta = 2\alpha$ nearly; then

$$\tau = T\frac{1}{\pi}(1-\rho)\frac{2\sin\frac{1}{2}\alpha}{\tan\frac{1}{2}\beta}. \tag{28}$$

If all the arms of the balance except P consist of resistance coils of very fine wire of no great length and doubled before being coiled, the induction coefficients belonging to these coils will be insensible, and τ will be reduced to $\frac{p}{P}$. The electric balance therefore affords the means of measuring the self-induction of any circuit whose resistance is known.

(46) It may also be used to determine the coefficient of induction between two circuits, as for instance, that between P and S which we have called m; but it would be more convenient to measure this by directly measuring the current, as in (37), without using the balance. We may also ascertain the equality of $\frac{p}{P}$ and $\frac{q}{Q}$ by there being no current of induction, and thus, when we know the value of p, we may determine that of q by a more perfect method than the comparison of deflections.

Exploration of the Electromagnetic Field

(47) Let us now suppose the primary circuit A to be of invariable form, and let us explore the electromagnetic field by means of the secondary circuit B, which we shall suppose to be variable in form and position.

We may begin by supposing B to consist of a short straight conductor with its extremities sliding on two parallel conducting rails, which are put in connection at some distance from the sliding-piece.

Then, if sliding the moveable conductor in a given direction increases the value of M, a negative electromotive force will act in the circuit B, tending to produce a negative current in B during the motion of the sliding-piece.

If a current be kept up in the circuit B, then the sliding-piece will

itself tend to move in that direction, which causes M to increase. At every point of the field there will always be a certain direction such that a conductor moved in that direction does not experience any electromotive force in whatever direction its extremities are turned. A conductor carrying a current will experience no mechanical force urging it in that direction or the opposite.

This direction is called the direction of the line of magnetic force through that point.

Motion of a conductor across such a line produces electromotive force in a direction perpendicular to the line and to the direction of motion, and a conductor carrying a current is urged in a direction perpendicular to the line and to the direction of the current.

(48) We may next suppose B to consist of a very small plane circuit capable of being placed in any position and of having its plane turned in any direction. The value of M will be greatest when the plane of the circuit is perpendicular to the line of magnetic force. Hence if a current is maintained in B it will tend to set itself in this position, and will of itself indicate, like a magnet, the direction of the magnetic force.

On Lines of Magnetic Force.

(49) Let any surface be drawn, cutting the lines of magnetic force, and on this surface let any system of lines be drawn at small intervals, so as to lie side by side without cutting each other. Next, let any line be drawn on the surface cutting all these lines, and let a second line be drawn near it, its distance from the first being such that the value of M for each of the small spaces enclosed between these two lines and the lines of the first system is equal to unity.

In this way let more lines be drawn so as to form a second system, so that the value of M for every reticulation formed by the intersection of the two systems of lines is unity.

Finally, from every point of intersection of these reticulations let a line be drawn through the field, always coinciding in direction with the direction of magnetic force.

(50) In this way the whole field will be filled with lines of magnetic force at regular intervals, and the properties of the electromagnetic field will be completely expressed by them.

For, 1st, If any closed curve be drawn in the field, the value of M for that curve will be expressed by the *number* of lines of force which *pass through* that closed curve.

2ndly. If this curve be a conducting circuit and be moved through

the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.

3rdly. If a current be maintained in the circuit, the conductor will be acted on by forces tending to move it so as to increase the number of lines passing through it, and the amount of work done by these forces is equal to the current in the circuit multiplied by the number of additional lines.

4thly. If a small plane circuit be placed in the field, and be free to turn, it will place its plane perpendicular to the lines of force. A small magnet will place itself with its axis in the direction of the lines of force.

5thly. If a long uniformly magnetized bar is placed in the field, each pole will be acted on by a force in the direction of the lines of force. The number of lines of force passing through unit of area is equal to the force acting on a unit pole multiplied by a coefficient depending on the magnetic nature of the medium, and called the coefficient of magnetic induction.

In fluids and isotropic solids the value of this coefficient μ is the same in whatever direction the lines of force pass through the substance, but in crystallized, strained, and organized solids the value of μ may depend on the direction of the lines of force with respect to the axes of crystallization, strain, or growth.

In all bodies μ is affected by temperature, and in iron it appears to diminish as the intensity of the magnetization increases.

On Magnetic Equipotential Surfaces

(51) If we explore the field with a uniformly magnetized bar, so long that one of its poles is in a very weak part of the magnetic field, then the magnetic forces will perform work on the other pole as it moves about the field.

If we start from a given point, and move this pole from it to any other point, the work performed will be independent of the path of the pole between the two points; provided that no electric current passes between the different paths pursued by the pole.

Hence, when there are no electric currents but only magnets in the field, we may draw a series of surfaces such that the work done in passing from one to another shall be constant whatever be the path pursued between them. Such surfaces are called Equipotential Surfaces, and in ordinary cases are perpendicular to the lines of magnetic force.

If these surfaces are so drawn that, when a unit pole passes from

any one to the next in order, unity of work is done, then the work done in any motion of a magnetic pole will be measured by the strength of the pole multiplied by the number of surfaces which it has passed through in the positive direction.

(52) If there are circuits carrying electric currents in the field, then there will still be equipotential surfaces in the parts of the field external to the conductors carrying the currents, but the work done on a unit pole in passing from one to another will depend on the number of times which the path of the pole circulates round any of these currents. Hence the potential in each surface will have a series of values in arithmetical progression, differing by the work done in passing completely round one of the currents in the field.

The equipotential surfaces will not be continuous closed surfaces, but some of them will be limited sheets, terminating in the electric circuit as their common edge or boundary. The number of these will be equal to the amount of work done on a unit pole in going round the current, and this by the ordinary measurement $= 4\pi\gamma$, where γ is the value of the current.

These surfaces, therefore, are connected with the electric current as soap-bubbles are connected with a ring in M. Plateau's experiments. Every current γ has $4\pi\gamma$ surfaces attached to it. These surfaces have the current for their common edge, and meet it at equal angles. The form of the surfaces in other parts depends on the presence of other currents and magnets, as well on the shape of the circuit to which they belong.

PART III

GENERAL EQUATIONS OF THE ELECTROMAGNETIC FIELD

(53) Let us assume three rectangular directions in space as the axes of x, y, and z, and let all quantities having direction be expressed by their components in these three directions.

Electrical Currents (p, q, r).

(54) An electrical current consists in the transmission of electricity from one part of a body to another. Let the quantity of electricity transmitted in unit of time cross unit of area perpendicular to the

axis of x be called p, then p is the component of the current at that place in the direction of x.

We shall use the letters p, q, r to denote the components of the current per unit of area in the directions of x, y, z.

Electrical Displacements (f, g, h).

(55) Electrical displacement consists in the opposite electrification of the sides of a molecule or particle of a body which may or may not be accompanied with transmission through the body. Let the quantity of electricity which would appear on the faces $dy.dz$ of an element dx, dy, dz cut from the body be $f.dy.dz$, then f is the component of electric displacement parallel to x. We shall use f, g, h to denote the electric displacements parallel to x, y, z respectively.

The variations of the electrical displacement must be added to the currents p, q, r to get the total motion of electricity, which we may call p', q', r', so that

$$\left.\begin{aligned} p' &= p + \frac{df}{dt} \\ q' &= q + \frac{dg}{dt} \\ r' &= r + \frac{dh}{dt} \end{aligned}\right\}. \qquad \text{(A)}$$

Electromotive Force (P, Q, R)

(56) Let P, Q, R represent the components of the electromotive force at any point. Then P represents the difference of potential per unit of length in a conductor placed in the direction of x at the given point. We may suppose an indefinitely short wire placed parallel to x at a given point and touched, during the action of the force P, by two small conductors, which are then insulated and removed from the influence of the electromotive force. The value of P might then be ascertained by measuring the charge of the conductors.

Thus if l be the length of the wire, the difference of potential at its ends will be Pl, and if C be the capacity of each of the small conductors the charge on each will be $\frac{1}{2}CPl$. Since the capacities of moderately large conductors, measured on the electromagnetic system, are exceedingly small, ordinary electromotive forces arising from electromagnetic actions could hardly be measured in this way.

In practice such measurements are always made with long conductors, forming closed or nearly closed circuits.

Electromagnetic Momentum (F, G, H)

(57) Let F, G, H represent the components of electromagnetic momentum at any point of the field, due to any system of magnets or currents.

Then F is the total impulse of the electromotive force in the direction of x that would be generated by the removal of these magnets or currents from the field, that is, if P be the electromotive force at any instant during the removal of the system

$$F=\int P\,dt.$$

Hence the part of the electromotive force which depends on the motion of magnets or currents in the field, or their alteration of intensity, is

$$P=-\frac{dF}{dt}, \qquad Q=-\frac{dG}{dt}, \qquad R=-\frac{dH}{dt}. \tag{29}$$

Electromagnetic Momentum of a Circuit

(58) Let s be the length of the circuit, then if we integrate

$$\int\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds \tag{30}$$

round the circuit, we shall get the total electromagnetic momentum of the circuit, or the number of lines of magnetic force which pass through it, the variations of which measure the total electromotive force in the circuit. This electromagnetic momentum is the same thing to which Professor Faraday has applied the name of the Electrotonic State.

If the circuit be the boundary of the elementary area $dy\,dz$, then its electromagnetic momentum is

$$\left(\frac{dH}{dy}-\frac{dG}{dz}\right)dy\,dz,$$

and this is the number of lines of magnetic force which pass through the area $dy\,dz$.

Magnetic Force (α, β, γ)

(59) Let α, β, γ represent the force acting on a unit magnetic pole placed at the given point resolved in the directions of x, y, and z.

Coefficient of Magnetic Induction (μ)

(60) Let μ be the ratio of the magnetic induction in a given medium to that in air under an equal magnetizing force, then the number of lines of force in unit of area perpendicular to x will be $\mu\alpha$ (μ is a quantity depending on the nature of the medium, its temperature, the amount of magnetization already produced, and in crystalline bodies varying with the direction).

(61) Expressing the electric momentum of small circuits perpendicular to the three axes in this notation, we obtain the following

Equations of Magnetic Force

$$\left.\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned}\right\}. \tag{B}$$

Equations of Currents

(62) It is known from experiment that the motion of a magnetic pole in the electromagnetic field in a closed circuit cannot generate work unless the circuit which the pole describes passes round an electric current. Hence, except in the space occupied by the electric currents,

$$\alpha dx + \beta dy + \gamma dz = d\phi \tag{31}$$

a complete differential of ϕ, the magnetic potential.

The quantity ϕ may be susceptible of an indefinite number of distinct values, according to the number of times that the exploring point passes round electric currents in its course, the difference between successive values of ϕ corresponding to a passage completely round a current of strength c being $4\pi c$.

Hence if there is no electric current,

$$\frac{d\gamma}{dy}-\frac{d\beta}{dz}=0;$$

but if there is a current p',

Similarly,

$$\left.\begin{aligned}\frac{d\gamma}{dy}-\frac{d\beta}{dz}&=4\pi p'\\ \frac{d\alpha}{dz}-\frac{d\gamma}{dx}&=4\pi q'\\ \frac{d\beta}{dx}-\frac{d\alpha}{dy}&=4\pi r'\end{aligned}\right\}. \tag{C}$$

We may call these the Equations of Currents.

Electromotive Force in a Circuit.

(63) Let ξ be the electromotive force acting round the circuit A, then

$$\xi=\int\left(P\frac{dx}{ds}+Q\frac{dy}{ds}+R\frac{dz}{ds}\right)ds, \tag{32}$$

where ds is the element of length, and the integration is performed round the circuit.

Let the forces in the field be those due to the circuits A and B, then the electromagnetic momentum of A is

$$\int\left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds=Lu+Mv, \tag{33}$$

where u and v are the currents in A and B, and

$$\xi=-\frac{d}{dt}(Lu+Mv). \tag{34}$$

Hence, if there is no motion of the circuit A,

$$\left.\begin{aligned}P&=-\frac{dF}{dt}-\frac{d\psi}{dx}\\ Q&=-\frac{dG}{dt}-\frac{d\psi}{dy}\\ R&=-\frac{dH}{dt}-\frac{d\psi}{dz}\end{aligned}\right\}, \tag{35}$$

where ψ is a function of x, y, z, and t, which is indeterminate as far as regards the solution of the above equations, because the terms depending on it will disappear on integrating round the circuit. The quantity ψ can always, however, be determined in any particular case when we know the actual conditions of the question. The physical interpretation of ψ is, that it represents the *electric potential at each point of space.*

Electromotive Force on a Moving Conductor

(64) Let a short straight conductor of length a, parallel to the axis of x, move with a velocity whose components are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and let its extremities slide along two parallel conductors with a velocity $\frac{ds}{dt}$. Let us find the alternation of the electromagnetic momentum of the circuit of which this arrangement forms a part.

In unit of time the moving conductor has travelled distances $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ along the directions of the three axes, and at the same time the lengths of the parallel conductors included in the circuit have each been increased by $\frac{ds}{dt}$.

Hence the quantity

$$\int \left(F\frac{dx}{ds}+G\frac{dy}{ds}+H\frac{dz}{ds}\right)ds$$

will be increased by the following increments,

$$a\left(\frac{dF}{dx}\frac{dx}{dt}+\frac{dF}{dy}\frac{dy}{dt}+\frac{dF}{dz}\frac{dz}{dt}\right), \text{ due to motion of conductor,}$$

$$-a\frac{ds}{dt}\left(\frac{dF}{dx}\frac{dx}{ds}+\frac{dG}{dx}\frac{dy}{ds}+\frac{dH}{dx}\frac{dz}{ds}\right), \text{ due to lengthening of circuit.}$$

The total increment will therefore be

$$a\left(\frac{dF}{dy}-\frac{dG}{dx}\right)\frac{dy}{dt}-a\left(\frac{dH}{dx}-\frac{dF}{dz}\right)\frac{dz}{dt};$$

or, by the equations of Magnetic Force (8),

$$-a\left(\mu\gamma\frac{dy}{dt}-\mu\beta\frac{dz}{dt}\right).$$

If P is the electromotive force in the moving conductor parallel to x referred to unit of length, then the actual electromotive force is Pa; and since this is measured by the decrement of the electromagnetic momentum of the circuit, the electromotive force due to motion will be

$$P = \mu\gamma\frac{dy}{dt} - \mu\beta\frac{dz}{dt}. \tag{36}$$

(65) The complete equations of electromotive force on a moving conductor may now be written as follows:—

Equations of Electromotive Force

$$\left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\psi}{dx} \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\psi}{dy} \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\psi}{dz} \end{aligned}\right\}. \tag{D}$$

The first term on the right-hand side of each equation represents the electromotive force arising from the motion of the conductor itself. This electromotive force is perpendicular to the direction of motion and to the lines of magnetic force; and if a parallelogram be drawn whose sides represent in direction and magnitude the velocity of the conductor and the magnetic induction at that point of the field, then the area of the parallelogram will represent the electromotive force due to the motion of the conductor, and the direction of the force is perpendicular to the plane of the parallelogram.

The second term in each equation indicates the effect of changes in the position or strength of magnets or currents in the field.

The third term shows the effect of the electric potential ψ. It has no effect in causing a circulating current in a closed circuit. It indicates the existence of a force urging the electricity to or from certain definite points in the field.

Electric Elasticity

(66) When an electromotive force acts on a dielectric, it puts every part of the dielectric into a polarized condition, in which its

opposite sides are oppositely electrified. The amount of this electrification depends on the electromotive force and on the nature of the substance, and, in solids having a structure defined by axes, on the direction of the electromotive force with respect to these axes. In isotropic substances, if k is the ratio of the electromotive force to the electric displacement, we may write the

Equations of Electric Elasticity

$$\left.\begin{aligned} P &= kf \\ Q &= kg \\ R &= kh \end{aligned}\right\}. \tag{E}$$

Electric Resistance

(67) When an electromotive force acts on a conductor it produces a current of electricity through it. This effect is additional to the electric displacement already considered. In solids of complex structure, the relation between the electromotive force and the current depends on their direction through the solid. In isotropic substances, which alone we shall here consider, if ρ is the specific resistance referred to unit of volume, we may write the

Equations of Electric Resistance

$$\left.\begin{aligned} P &= -\rho p \\ Q &= -\rho q \\ R &= -\rho r \end{aligned}\right\} \tag{F}$$

Electric Quantity

(68) Let e represent the quantity of free positive electricity contained in unit of volume at any part of the field, then, since this arises from the electrification of the different parts of the field not neutralizing each other, we may write the

Equation of Free Electricity

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0. \tag{G}$$

(69) If the medium conducts electricity, then we shall have another condition, which may be called, as in hydrodynamics, the

Equation of Continuity

$$\frac{de}{dt}+\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}=0. \qquad \text{(H)}$$

(70) In these equations of the electromagnetic field we have assumed twenty variable quantities, namely,

For Electromagnetic Momentum	$F \quad G \quad H$
For Magnetic Intensity	$\alpha \quad \beta \quad \gamma$
For Electromotive Force	$P \quad Q \quad R$
For Current due to true Conduction	$p \quad q \quad r$
For Electric Displacement	$f \quad g \quad h$
For Total Current (including variation of displacement)	$p' \quad q' \quad r'$
For Quantity of Free Electricity	e
For Electric Potential	Ψ

Between these twenty quantities we have found twenty equations, viz.

Three equations of Magnetic Force	(B)
Three equations of Electric Currents	(C)
Three equations of Electromotive Force	(D)
Three equations of Electric Elasticity	(E)
Three equations of Electric Resistance	(F)
Three equations of Total Currents	(A)
One equation of Free Electricity	(G)
One equation of Continuity	(H)

These equations are therefore sufficient to determine all the quantities which occur in them, provided we know the conditions of the problem. In many questions, however, only a few of the equations are required.

Intrinsic Energy of the Electromagnetic Field

(71) We have seen (33) that the intrinsic energy of any system of currents is found by multiplying half the current in each circuit into its electromagnetic momentum. This is equivalent to finding the integral

$$E=\tfrac{1}{2}\sum(Fp'+Gq'+Hr')\,dV \qquad (37)$$

over all the space occupied by currents, where p, q, r are the

components of currents, and F, G, H the components of electromagnetic momentum.

Substituting the values of p', q', r' from the equations of Currents (C), this becomes

$$\frac{1}{8\pi}\sum\left\{F\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)+G\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right)+H\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\right\}dV.$$

Integrating by parts, and remembering that α, β, γ vanish at an infinite distance, the expression becomes

$$\frac{1}{8\pi}\sum\left\{\alpha\left(\frac{dH}{dy}-\frac{dG}{dz}\right)+\beta\left(\frac{dF}{dz}-\frac{dH}{dx}\right)+\gamma\left(\frac{dG}{dx}-\frac{dF}{dy}\right)\right\}dV,$$

where the integration is to be extended over all space. Referring to the equations of Magnetic Force (B), p. 63, this becomes

$$E=\frac{1}{8\pi}\sum\{\alpha\,.\,\mu\alpha+\beta\,.\,\mu\beta+\gamma\,.\,\mu\gamma\}\,dV \tag{38}$$

where α, β, γ are the components of magnetic intensity or the force on a unit magnetic pole, and $\mu\alpha$, $\mu\beta$, $\mu\gamma$ are the components of the quantity of magnetic induction, or the number of lines of force in unit of area.

In isotropic media the value of μ is the same in all directions, and we may express the result more simply by saying that the intrinsic energy of any part of the magnetic field arising from its magnetization is

$$\frac{\mu}{8\pi}I^2$$

per unit of volume, where I is the magnetic intensity.

(72) Energy may be stored up in the field in a different way, namely, by the action of electromotive force in producing electric displacement. The work done by a variable electromotive force, P, in producing a variable displacement, f, is got by integrating

$$\int Pdf$$

from $P=0$ to the given value of P.

Since $P=kf$, equation (E), this quantity becomes

$$\int kfdf=\tfrac{1}{2}kf^2=\tfrac{1}{2}Pf.$$

Hence the intrinsic energy of any part of the field, as existing in the form of electric displacement, is

$$\tfrac{1}{2}\sum(Pf+Qg+Rh)\,dV.$$

The total energy existing in the field is therefore

$$E=\sum\left\{\frac{1}{8\pi}(\alpha\mu\alpha+\beta\mu\beta+\gamma\mu\gamma)+\tfrac{1}{2}(Pf+Qg+Rh)\right\}dV. \qquad \text{(I)}$$

The first term of this expression depends on the magnetization of the field, and is explained on our theory by actual motion of some kind. The second term depends on the electric polarization of the field, and is explained on our theory by strain of some kind in an elastic medium.

(73) I have on a former occasion* attempted to describe a particular kind of motion and a particular kind of strain, so arranged as to account for the phenomena. In the present paper I avoid any hypothesis of this kind; and in using such words as electric momentum and electric elasticity in reference to the known phenomena of the induction of currents and the polarization of dielectrics, I wish merely to direct the mind of the reader to mechanical phenomena which will assist him in understanding the electrical ones. All such phrases in the present paper are to be considered as illustrative, not as explanatory.

(74) In speaking of the Energy of the field, however, I wish to be understood literally. All energy is the same as mechanical energy, whether it exists in the form of motion or in that of elasticity, or in any other form. The energy in electromagnetic phenomena is mechanical energy. The only question is, Where does it reside? On the old theories it resides in the electrified bodies, conducting circuits, and magnets, in the form of an unknown quality called potential energy, or the power of producing certain effects at a distance. On our theory it resides in the electromagnetic field, in the space surrounding the electrified and magnetic bodies, as well as in those bodies themselves, and is in two different forms, which may be described without hypothesis as magnetic polarization and electric polarization, or, according to a very probable hypothesis, as the motion and the strain of one and the same medium.

* "On Physical Lines of Force", *Philosophical Magazine*, 1861–62.

(75) The conclusions arrived at in the present paper are independent of this hypothesis, being deduced from experimental facts of three kinds:

1. The induction of electric currents by the increase or diminution of neighbouring currents according to the changes in the lines of force passing through the circuit.
2. The distribution of magnetic intensity according to the variations of a magnetic potential.
3. The induction (or influence) of statical electricity through dielectrics.

We may now proceed to demonstrate from these principles the existence and laws of the mechanical forces which act upon electric currents, magnets, and electrified bodies placed in the electromagnetic field.

PART IV

MECHANICAL ACTIONS IN THE FIELD

Mechanical Force on a Moveable Conductor

(76) We have shown (§§ 34 & 35) that the work done by the electromagnetic forces in aiding the motion of a conductor is equal to the product of the current in the conductor multiplied by the increment of the electromagnetic momentum due to the motion.

Let a short straight conductor of length a move parallel to itself in the direction of x, with its extremities on two parallel conductors. Then the increment of the electromagnetic momentum due to the motion of a will be

$$a\left(\frac{dF}{dx}\frac{dx}{ds}+\frac{dG}{dx}\frac{dy}{ds}+\frac{dH}{dx}\frac{dz}{ds}\right)\delta x.$$

That due to the lengthening of the circuit by increasing the length of the parallel conductors will be

$$-a\left(\frac{dF}{dx}\frac{dx}{ds}+\frac{dF}{dy}\frac{dy}{ds}+\frac{dF}{dz}\frac{dz}{ds}\right)\delta x.$$

The total increment is

$$a\delta x\left\{\frac{dy}{ds}\left(\frac{dG}{dx}-\frac{dF}{dy}\right)-\frac{dz}{ds}\left(\frac{dF}{dz}-\frac{dH}{dx}\right)\right\},$$

which is by the equations of Magnetic Force (B), p. 63,

$$a\delta x\left(\frac{dy}{ds}\mu\gamma-\frac{dz}{ds}\mu\beta\right).$$

Let X be the force acting along the direction of x per unit of length of the conductor, then the work done is $Xa\delta x$.

Let C be the current in the conductor, and let p', q', r' be its components, then

$$Xa\delta x=Ca\delta x\left(\frac{dy}{ds}\mu\gamma-\frac{dz}{ds}\mu\beta\right),$$

or

Similarly,

$$\left.\begin{aligned}X&=\mu\gamma q'-\mu\beta r'\\ Y&=\mu\alpha r'-\mu\gamma p'\\ Z&=\mu\beta p'-\mu\alpha q'\end{aligned}\right\}. \tag{J}$$

These are the equations which determine the mechanical force acting on a conductor carrying a current. The force is perpendicular to the current and to the lines of force, and is measured by the area of the parallelogram formed by lines parallel to the current and lines of force, and proportional to their intensities.

Mechanical Force on a Magnet

(77) In any part of the field not traversed by electric currents the distribution of magnetic intensity may be represented by the differential coefficients of a function which may be called the magnetic potential. When there are no currents in the field, this quantity has a single value for each point. When there are currents, the potential has a series of values at each point, but its differential coefficients have only one value, namely,

$$\frac{d\phi}{dx}=\alpha,\qquad \frac{d\phi}{dy}=\beta,\qquad \frac{d\phi}{dz}=\gamma.$$

Substituting these values of α, β, γ in the expression (equation (38)) for the intrinsic energy of the field, and integrating by parts, it becomes

$$-\sum\left\{\phi\frac{1}{8\pi}\left(\frac{d\mu\alpha}{dx}+\frac{d\mu\beta}{dy}+\frac{d\mu\gamma}{dz}\right)\right\}dV.$$

The expression

$$\sum\left(\frac{d\mu\alpha}{dx}+\frac{d\mu\beta}{dy}+\frac{d\mu\gamma}{dz}\right)dV=\sum m\,dV \tag{39}$$

indicates the number of lines of magnetic force which have their origin within the space V. Now a magnetic pole is known to us only as the origin or termination of lines of magnetic force, and a unit pole is one which has 4π lines belonging to it, since it produces unit of magnetic intensity at unit of distance over a sphere whose surface is 4π.

Hence if m is the amount of free positive magnetism in unit of volume, the above expression may be written $4\pi m$, and the expression for the energy of the field becomes

$$E = -\sum (\tfrac{1}{2}\phi m)\, dV. \tag{40}$$

If there are two magnetic poles m_1 and m_2 producing potentials ϕ_1 and ϕ_2 in field, then if m_2 is moved a distance dx, and is urged in that direction by a force X, then the work done is $X\,dx$, and the decrease of energy in the field is

$$d\{\tfrac{1}{2}(\phi_1+\phi_2)(m_1+m_2)\},$$

and these must be equal by the principle of Conservation of Energy.

Since the distribution ϕ_1 is determined by m_1, and ϕ_2 by m_2, the quantities $\phi_1 m_1$ and $\phi_2 m_2$ will remain constant.

It can be shown also, as Green has proved (*On the Application of Mathematical Analysis to Electricity*, p. 10), that

$$m_1\phi_2 = m_2\phi_1,$$

so that we get $\quad X\,dx = d(m_2\phi_1),$

or

$$\left.\begin{aligned} X &= m_2\frac{d\phi_1}{dx} = m_2\alpha_1, \\ &\text{where } \alpha_1 \text{ represents the magnetic intensity due to } m_1. \\ &\text{Similarly,} \quad Y = m_2\beta_1, \\ Z &= m_2\gamma_1. \end{aligned}\right\} \tag{K}$$

So that a magnetic pole is urged in the direction of the lines of magnetic force with a force equal to the product of the strength of the pole and the magnetic intensity.

(78) If a single magnetic pole, that is, one pole of a very long magnet, be placed in the field, the only solution of ϕ is

$$\phi_1 = -\frac{m_1}{\mu}\frac{1}{r}, \tag{41}$$

where m_1 is the strength of the pole, and r the distance from it.

The repulsion between two poles of strength m_1 and m_2 is

$$m_2 \frac{d\phi_1}{dr} = \frac{m_1 m_2}{\mu r^2}. \tag{42}$$

In air or any medium in which $\mu = 1$ this is simply $\frac{m_1 m_2}{r^2}$, but in other media the force acting between two given magnetic poles is inversely proportional to the coefficient of magnetic induction for the medium. This may be explained by the magnetization of the medium induced by the action of the poles.

Mechanical Force on an Electrified Body

(79) If there is no motion or change of strength of currents or magnets in the field, the electromotive force is entirely due to variation of electric potential, and we shall have (§ 65)

$$P = -\frac{d\Psi}{dx}, \qquad Q = -\frac{d\Psi}{dy}, \qquad R = -\frac{d\Psi}{dz}.$$

Integrating by parts the expression (I) for the energy due to electric displacement, and remembering that P, Q, R vanish at an infinite distance, it becomes

$$\tfrac{1}{2}\sum\left\{\Psi\left(\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}\right)dV,\right.$$

or by the equation of Free Electricity (G), p. 67,

$$-\tfrac{1}{2}\sum(\Psi e)\, dV.$$

By the same demonstration as was used in the case of the mechanical action on a magnet, it may be shown that the mechanical force on a small body containing a quantity e_2 of free electricity placed in a field whose potential arising from other electrified bodies is Ψ_1, has for components

$$\left.\begin{aligned} X &= e_2 \frac{d\Psi_1}{dx} = -P_1 e_2 \\ Y &= e_2 \frac{d\Psi_1}{dy} = -Q_1 e_2 \\ Z &= e_3 \frac{d\Psi_1}{dz} = -R_1 e_2 \end{aligned}\right\}. \tag{D}$$

So that an electrified body is urged in the direction of the electromotive force with a force equal to the product of the quantity of free electricity and the electromotive force.

If the electrification of the field arises from the presence of a small electrified body containing e_1 of free electricity, the only solution of Ψ_1 is

$$\Psi_1 = \frac{k}{4\pi}\frac{e_1}{r}, \tag{43}$$

where r is the distance from the electrified body.

The repulsion between two electrified bodies e_1, e_2 is therefore

$$e_2 \frac{d\Psi_1}{dr} = \frac{k}{4\pi}\frac{e_1 e_2}{r^2}. \tag{44}$$

Measurement of Electrostatic Effects

(80) The quantities with which we have had to do have been hitherto expressed in terms of the Electromagnetic System of measurement, which is founded on the mechanical action between currents. The electrostatic system of measurement is founded on the mechanical action between electrified bodies, and is independent of, and incompatible with, the electromagnetic system; so that the units of the different kinds of quantity have different values according to the system we adopt, and to pass from the one system to the other, a reduction of all the quantities is required.

According to the electrostatic system, the repulsion between two small bodies charged with quantities η_1, η_2 of electricity is

$$\frac{\eta_1 \eta_2}{r^2},$$

where r is the distance between them.

Let the relation of the two systems be such that one electromagnetic unit of electricity contains v electrostatic units; then $\eta_1 = ve_1$ and $\eta_2 = ve_2$, and this repulsion becomes

$$v^2 \frac{e_1 e_2}{r^2} = \frac{k}{4\pi}\frac{e_1 e_2}{r^2} \quad \text{by equation (44)}, \tag{45}$$

whence k, the coefficient of "electric elasticity" in the medium in which the experiments are made, *i.e.* common air, is related to v, the number of electrostatic units in one electromagnetic unit, by the equation

$$k = 4\pi v^2. \tag{46}$$

The quantity v may be determined by experiment in several ways. According to the experiments of MM. Weber and Kohlrausch,

$$v = 310{,}740{,}000 \text{ metres per second.}$$

(81) It appears from this investigation, that if we assume that the medium which constitutes the electromagnetic field is, when dielectric, capable of receiving in every part of it an electric polarization, in which the opposite sides of every element into which we may conceive the medium divided are oppositely electrified, and if we also assume that this polarization or electric displacement is proportional to the electromotive force which produces or maintains it, then we can show that electrified bodies in a dielectric medium will act on one another with forces obeying the same laws as are established by experiment.

The energy, by the expenditure of which electrical attractions and repulsions are produced, we suppose to be stored up in the dielectric medium which surrounds the electrified bodies, and not on the surface of those bodies themselves, which on our theory are merely the bounding surfaces of the air or other dielectric in which the true springs of action are to be sought.

Note on the Attraction of Gravitation

(82) After tracing to the action of the surrounding medium both the magnetic and the electric attractions and repulsions, and finding them to depend on the inverse square of the distance, we are naturally led to inquire whether the attraction of gravitation, which follows the same law of the distance, is not also traceable to the action of a surrounding medium.

Gravitation differs from magnetism and electricity in this; that the bodies concerned are all of the same kind, instead of being of opposite signs, like magnetic poles and electrified bodies, and that the force between these bodies is an attraction and not a repulsion, as is the case between like electric and magnetic bodies.

The lines of gravitating force near two dense bodies are exactly of the same form as the lines of magnetic force near two poles of the same name; but whereas the poles are repelled, the bodies are attracted. Let E be the intrinsic energy of the field surrounding two gravitating bodies M_1, M_2, and let E' be the intrinsic energy of the field surrounding two magnetic poles, m_1, m_2 equal in numerical value to M_1, M_2, and let X be the gravitating force acting during the

displacement δx, and X' the magnetic force,

$$X\,\delta x = \delta E, \qquad X'\,\delta x = \delta E';$$

now X and X' are equal in numerical value, but of opposite signs; so that

$$\delta E = -\delta E',$$

or

$$\begin{aligned} E &= C - E' \\ &= C - \sum \frac{1}{8\pi}(\alpha^2+\beta^2+\gamma^2)\,dV, \end{aligned}$$

where α, β, γ are the components of magnetic intensity. If R be the resultant gravitating force, and R' the resultant magnetic force at a corresponding part of the field,

$$R = -R', \quad \text{and} \quad \alpha^2+\beta^2+\gamma^2 = R^2 = R'^2.$$

Hence

$$E = C - \sum \frac{1}{8\pi} R^2\,dV. \tag{47}$$

The intrinsic energy of the field of gravitation must therefore be less wherever there is a resultant gravitating force.

As energy is essentially positive, it is impossible for any part of space to have negative intrinsic energy. Hence those parts of space in which there is no resultant force, such as the points of equilibrium in the space between the different bodies of a system, and within the substance of each body, must have an intrinsic energy per unit of volume greater than

$$\frac{1}{8\pi} R^2,$$

where R is the greatest possible value of the intensity of gravitating force in any part of the universe.

The assumption, therefore, that gravitation arises from the action of the surrounding medium in the way pointed out, leads to the conclusion that every part of this medium possesses, when undisturbed, an enormous intrinsic energy, and that the presence of dense bodies influences the medium so as to diminish this energy wherever there is a resultant attraction.

As I am unable to understand in what way a medium can possess such properties, I cannot go any further in this direction in searching for the cause of gravitation.

PART V

THEORY OF CONDENSERS

Capacity of a Condenser

(83) The simplest form of condenser consists of a uniform layer of insulating matter bounded by two conducting surfaces, and its capacity is measured by the quantity of electricity on either surface when the difference of potentials is unity.

Let S be the area of either surface, a the thickness of the dielectric, and k its coefficient of electric elasticity; then on one side of the condenser the potential is Ψ_1, and on the other side Ψ_1+1, and within its substance

$$\frac{d\Psi}{dx}=\frac{1}{a}=kf. \tag{48}$$

Since $\frac{d\Psi}{dx}$ and therefore f is zero outside the condenser, the quantity of electricity on its first surface $=-Sf$, and on the second $+Sf$. The capacity of the condenser is therefore $Sf=\frac{S}{ak}$ in electromagnetic measure.

Specific Capacity of Electric Induction (D)

(84) If the dielectric of the condenser be air, then its capacity in electrostatic measure is $\frac{S}{4\pi a}$ (neglecting corrections arising from the conditions to be fulfilled at the edges). If the dielectric have a capacity whose ratio to that of air is D, then the capacity of the condenser will be $\frac{DS}{4\pi a}$.

Hence
$$D=\frac{k_0}{k}, \tag{49}$$

where k_0 is the value of k in air, which is taken for unity.

Electric Absorption

(85) When the dielectric of which the condenser is formed is not a perfect insulator, the phenomena of conduction are combined with those of electric displacement. The condenser, when left charged, gradually loses its charge, and in some cases, after being discharged

completely, it gradually acquires a new charge of the same sign as the original charge, and this finally disappears. These phenomena have been described by Professor Faraday (*Experimental Researches*, Series XI.) and by Mr F. Jenkin (*Report of Committee of Board of Trade on Submarine Cables*), and may be classed under the name of "Electric Absorption".

(86) We shall take the case of a condenser composed of any number of parallel layers of different materials. If a constant difference of potentials between its extreme surfaces is kept up for a sufficient time till a condition of permanent steady flow of electricity is established, then each bounding surface will have a charge of electricity depending on the nature of the substances on each side of it. If the extreme surfaces be now discharged, these internal charges will gradually be dissipated, and a certain charge may reappear on the extreme surfaces if they are insulated, or, if they are connected by a conductor, a certain quantity of electricity may be urged through the conductor during the re-establishment of equilibrium.

Let the thickness of the several layers of the condenser be a_1, a_2, &c.

Let the values of k for these layers be respectively k_1, k_2, k_3 and let

$$a_1 k_2 + a_2 k_2 + \&\text{c.} = ak, \tag{50}$$

where k is the "electric elasticity" of air, and a is the thickness of an equivalent condenser of air.

Let the resistances of the layers be respectively r_1, r_2, &c., and let $r_1 + r_2 + \&\text{c.} = r$ be the resistance of the whole condenser, to a steady current through it per unit of surface.

Let electric displacement in each layer be f_1, f_2, &c.

Let the electric current in each layer be p_1, p_2, &c.

Let the potential on the first surface be Ψ_1, and the electricity per unit of surface e_1.

Let the corresponding quantities at the boundary of the first and second surface be Ψ_2 and e_2, and so on. Then by equations (G) and (H),

$$\left.\begin{array}{ll} e_1 = -f_1, & \dfrac{de_1}{dt} = -p_1, \\[2ex] e_2 = f_1 - f_2, & \dfrac{de_2}{dt} = p_1 - p_2, \\[2ex] \&\text{c.} & \&\text{c.} \end{array}\right\}. \tag{51}$$

But by equations (E) and (F)

$$\left.\begin{array}{l}\Psi_1-\Psi_2=a_1k_1f_1=-r_1p_1\\ \Psi_2-\Psi_3=a_2k_2f_2=-r_2p_2,\\ \quad \&c. \qquad \&c. \qquad \&c.\end{array}\right\}. \tag{52}$$

After the electromotive force has been kept up for a sufficient time the current becomes the same in each layer, and

$$p_1=p_2=\&c.=p=\frac{\Psi}{r},$$

where Ψ is the total difference of potentials between the extreme layers. We have then

$$\left.\begin{array}{lll} & f_1=-\dfrac{\Psi}{r}\dfrac{r_1}{a_1k_1}, & f_2=-\dfrac{\Psi}{r}\dfrac{r_2}{a_2k_2}, \&c. \\ \text{and} & e_1=\dfrac{\Psi}{r}\dfrac{r_1}{a_1k_1}, & e_2=\dfrac{\Psi}{r}\left(\dfrac{r_2}{a_2k_2}-\dfrac{r_1}{a_1k_1}\right), \&c.\end{array}\right\} \tag{53}$$

These are the quantities of electricity on the different surfaces.

(87) Now let the condenser be discharged by connecting the extreme surfaces through a perfect conductor so that their potentials are instantly rendered equal, then the electricity on the extreme surfaces will be altered, but that on the internal surfaces will not have time to escape. The total difference of potentials is now

$$\Psi'=a_1k_1e_1'+a_2k_2(e_1'+e_2)+a_3k_3(e_1'+e_2+e_3), \qquad \&c.=0, \tag{54}$$

whence if e_1' is what e_1 becomes at the instant of discharge,

$$e_1'=\frac{\Psi}{r}\frac{r_1}{a_1k_1}-\frac{\Psi}{ak}=e_1-\frac{\Psi}{ak}. \tag{55}$$

The instantaneous discharge is therefore $\dfrac{\Psi}{ak}$, or the quantity which would be discharged by a condenser of air of the equivalent thickness a and it is unaffected by the want of perfect insulation.

(88) Now let us suppose the connection between the extreme surfaces broken, and the condenser left to itself, and let us consider the gradual dissipation of the internal charges. Let Ψ' be the

difference of potential of the extreme surfaces at any time t; then

$$\Psi' = a_1 k_1 f_1 + a_2 k_2 f_2 + \&c; \tag{56}$$

but

$$a_1 k_1 f_1 = -r_1 \frac{df_1}{dt},$$

$$a_2 k_2 f_2 = -r_2 \frac{df_2}{dt}.$$

Hence $f_1 = A_1 e - \dfrac{a_1 k_1}{r_1} t$, $f_2 = A_2 e - \dfrac{a_2 k_2}{r_2} t$, &c.; and by referring to the values of e_1', e_2, &c., we find

$$\left.\begin{aligned} A_1 &= \frac{\Psi}{r}\frac{r_1}{a_1 k_1} - \frac{\Psi}{ak} \\ A_2 &= \frac{\Psi}{r}\frac{r_2}{a_2 k_2} - \frac{\Psi}{ak} \\ &\&c. \end{aligned}\right\}, \tag{57}$$

so that we find for the difference of extreme potentials at any time,

$$\Psi' = \Psi\left\{\left(\frac{r_1}{r}\frac{a_1 k_1}{ak}\right) e - \frac{a_1 k_1}{r_1} t + \left(\frac{r_2}{r}\frac{a_2 k_2}{ak}\right) e - \frac{a_2 k_2}{r_2} t + \&c. \right. \tag{58}$$

(89) It appears from this result that if all the layers are made of the same substance, Ψ' will be zero always. If they are of different substances, the order in which they are placed is indifferent, and the effect will be the same whether each substance consists of one layer, or is divided into any number of thin layers and arranged in any order among thin layers of the other substances. Any substance, therefore, the parts of which are not mathematically homogeneous, though they may be apparently so, may exhibit phenomena of absorption. Also, since the order of magnitude of the coefficients is the same as that of the indices, the value of Ψ' can never change sign, but must start from zero, become positive, and finally disappear.

(90) Let us next consider the total amount of electricity which would pass from the first surface to the second, if the condenser, after being thoroughly saturated by the current and then discharged, has its extreme surfaces connected by a conductor of resistance R. Let p be the current in this conductor; then, during the discharge,

$$\Psi' = p_1 r_1 + p_2 r_2 + \&c. = pR. \tag{59}$$

Integrating with respect to the time, and calling q_1, q_2, q the quantities of electricity which traverse the different conductors,

$$q_1 r_1 + q_2 r_2 + \&c. = qR. \tag{60}$$

The quantities of electricity on the several surfaces will be

$$\begin{aligned} &e'_1 - q - q_1, \\ &e_2 + q_1 - q_2, \\ &\&c.; \end{aligned}$$

and since at last all these quantities vanish, we find

$$\begin{aligned} &q_1 - e'_1 - q, \\ &q_2 = e'_1 + e_2 - q; \end{aligned}$$

whence
$$qR = \frac{\Psi}{r}\left(\frac{r_1^2}{a_1 k_1} + \frac{r_2^2}{a_2 k_2} + \&c.\right) - \frac{\Psi r}{ak},$$

or

$$q = \frac{\Psi}{akrR}\left\{a_1 k_1 a_2 k_2 \left(\frac{r_1}{a_1 k_1} - \frac{r_2}{a_2 k_2}\right)^2 \right.$$
$$\left. + a_2 k_2 a_3 k_3 \left(\frac{r_2}{a_2 k_2} - \frac{r_3}{a_3 k_3}\right)^3 + \&c.\right\}, \tag{61}$$

a quantity essentially positive; so that, when the primary electrification is in one direction, the secondary discharge is always in the same direction as the primary discharge*.

PART VI

ELECTROMAGNETIC THEORY OF LIGHT

(91) At the commencement of this paper we made use of the optical hypothesis of an elastic medium through which the vibrations of light are propagated, in order to show that we have warrantable grounds for seeking, in the same medium, the cause of other

* Since this paper was communicated to the Royal Society, I have seen a paper by M. Gaugain in the *Annales de Chimie* for 1864, in which he has deduced the phenomena of electric absorption and secondary discharge from the theory of compound condensers.

phenomena as well as those of light. We then examined electromagnetic phenomena, seeking for their explanation in the properties of the field which surrounds the electrified or magnetic bodies. In this way we arrived at certain equations expressing certain properties of the electromagnetic field. We now proceed to investigate whether these properties of that which constitutes the electromagnetic field, deduced from electromagnetic phenomena alone, are sufficient to explain the propagation of light through the same substance.

(92) Let us suppose that a plane wave whose direction cosines are l, m, n is propagated through the field with a velocity V. Then all the electromagnetic functions will be functions of

$$w = lx + my + nz - Vt.$$

The equations of Magnetic Force (B), p. 63, will become

$$\mu\alpha = m\frac{dH}{dw} - n\frac{dG}{dw},$$

$$\mu\beta = n\frac{dF}{dw} - l\frac{dH}{dw},$$

$$\mu\gamma = l\frac{dG}{dw} - m\frac{dF}{dw}.$$

If we multiply these equations respectively by l, m, n, and add, we find

$$l\mu\alpha + m\mu\beta + n\mu\gamma = 0, \tag{62}$$

which shows that the direction of the magnetization must be in the plane of the wave.

(93) If we combine the equations of Magnetic Force (B) with those of Electric Currents (C), and put for brevity

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = J, \quad \text{and} \quad \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = \nabla^2, \tag{63}$$

$\nabla \cdot A$

$- \nabla^2 A$

$$\left.\begin{aligned} 4\pi\mu p' &= \frac{dJ}{dx} - \nabla^2 F \\ 4\pi\mu q' &= \frac{dJ}{dy} - \nabla^2 G \\ 4\pi\mu r' &= \frac{dJ}{dz} - \nabla^2 H \end{aligned}\right\}. \tag{64}$$

If the medium in the field is a perfect dielectric there is no true conduction, and the currents p', q', r' are only variations in the electric displacement, or, by the equations of Total Currents (A),

$$p' = \frac{df}{dt}, \qquad q' = \frac{dg}{dt}, \qquad r' = \frac{dh}{dt}. \tag{65}$$

But these electric displacements are caused by electromotive forces, and by the equations of Electric Elasticity (E),

$$P = kf, \qquad Q = kg, \qquad R = kh. \tag{66}$$

These electromotive forces are due to the variations either of the electromagnetic or the electrostatic functions, as there is no motion of conductors in the field; so that the equations of electromotive force (D) are

$\vec{E} = -\frac{dA}{dt} - \nabla\phi$

$$\left.\begin{aligned} P &= -\frac{dF}{dt} - \frac{d\Psi}{dx} \\ Q &= -\frac{dG}{dt} - \frac{d\Psi}{dy} \\ R &= -\frac{dH}{dt} - \frac{d\Psi}{dz} \end{aligned}\right\}. \tag{67}$$

(94) Combining these equations, we obtain the following:—

$$\left.\begin{aligned} k\left(\frac{dJ}{dx} - \nabla^2 F\right) + 4\pi\mu\left(\frac{d^2F}{dt^2} + \frac{d^2\Psi}{dxdt}\right) &= 0 \\ k\left(\frac{dJ}{dy} - \nabla^2 G\right) + 4\pi\mu\left(\frac{d^2G}{dt^2} + \frac{d^2\Psi}{dydt}\right) &= 0 \\ k\left(\frac{dJ}{dz} - \nabla^2 H\right) + 4\pi\mu\left(\frac{d^2H}{dt^2} + \frac{d^2\Psi}{dzdt}\right) &= 0 \end{aligned}\right\}. \tag{68}$$

If we differentiate the third of these equations with respect to y, and the second with respect to z, and subtract, J and Ψ disappear, and by remembering the equations (B) of magnetic force, the results may be written

$$\left.\begin{aligned} k\nabla^2\mu\alpha &= 4\pi\mu\frac{d^2}{dt^2}\mu\alpha \\ k\nabla^2\mu\beta &= 4\pi\mu\frac{d^2}{dt^2}\mu\beta \\ k\nabla^2\mu\gamma &= 4\pi\mu\frac{d^2}{dt^2}\mu\gamma \end{aligned}\right\}. \tag{69}$$

(95) If we assume that α, β, γ are functions of $lx+my+nz-Vt=w$, the first equation becomes

$$k\mu\frac{d^2\alpha}{dw^2}=4\pi\mu^2V^2\frac{d^2\alpha}{dw^2}, \tag{70}$$

or

$$V=\pm\sqrt{\frac{k}{4\pi\mu}}. \tag{71}$$

The other equations give the same value for V, so that the wave is propagated in either direction with a velocity V.

This wave consists entirely of magnetic disturbances, the direction of magnetization being in the plane of the wave. No magnetic disturbance whose direction of magnetization is not in the plane of the wave can be propagated as a plane wave at all.

Hence magnetic disturbances propagated through the electromagnetic field agree with light in this, that the disturbance at any point is transverse to the direction of propagation, and such waves may have all the properties of polarized light.

(96) The only medium in which experiments have been made to determine the value of k is air, in which $\mu=1$, and therefore, by equation (46),

$$V=v. \tag{72}$$

By the electromagnetic experiments of MM. Weber and Kohlrausch*,

$$v=310{,}740{,}000 \text{ metres per second}$$

is the number of electrostatic units in one electromagnetic unit of electricity, and this, according to our result, should be equal to the velocity of light in air or vacuum.

The velocity of light in air, by M. Fizeau's† experiments, is

$$V=314{,}858{,}000;$$

according to the more accurate experiments of M. Foucault‡,

$$V=298{,}000{,}000.$$

The velocity of light in the space surrounding the earth, deduced from the coefficient of aberration and the received value of the

* *Leipzig Transactions*, Vol. v. (1857), p. 260, or Poggendorff's *Annalen*, Aug. 1856, p. 10.

† *Comptes Rendus*, Vol. XXIX. (1849), p. 90.

‡ *Ibid.*, Vol. LV. (1862), pp. 501, 792.

radius of the earth's orbit, is

$$V = 308{,}000{,}000.$$

(97) Hence the velocity of light deduced from experiment agrees sufficiently well with the value of v deduced from the only set of experiments we as yet possess. The value of v was determined by measuring the electromotive force with which a condenser of known capacity was charged, and then discharging the condenser through a galvanometer, so as to measure the quantity of electricity in it in electromagnetic measure. The only use made of light in the experiment was to see the instruments. The value of V found by M. Foucault was obtained by determining the angle through which a revolving mirror turned, while the light reflected from it went and returned along a measured course. No use whatever was made of electricity or magnetism.

The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.

(98) Let us now go back upon the equations in (94), in which the quantities J and Ψ occur, to see whether any other kind of disturbance can be propagated through the medium depending on these quantities which disappeared from the final equations.

If we determine χ from the equation

$\nabla\cdot\nabla\chi$, $J = \nabla\cdot A$

$\nabla\chi$

$$\nabla^2\chi = \frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} + \frac{d^2\chi}{dz^2} = J, \tag{73}$$

and F', G', H' from the equations

$A - \nabla\chi$

$$F' = F \; \frac{d\chi}{dx}, \qquad G' - G - \frac{d\chi}{dy}, \qquad H' = H - \frac{d\chi}{dz}, \tag{74}$$

then $\nabla\cdot A = 0$

$$\frac{dF'}{dx} + \frac{dG'}{dy} + \frac{dH'}{dz} = 0, \tag{75}$$

and the equations in (94) become of the form

$$k\nabla^2 F' = 4\pi\mu\left\{\frac{d^2F'}{dt^2} + \frac{d}{dx\,dt}\left(\Psi + \frac{d\chi}{dt}\right)\right\}. \tag{76}$$

Differentiating the three equations with respect to x, y, and z, and adding, we find that

$\Psi = -\frac{d\chi}{dt} + \ldots y, z)$

$$\Psi = -\frac{d\chi}{dt} + \phi(x, y, z), \tag{77}$$

and that

$$\left.\begin{aligned} k\nabla^2 F' &= 4\pi\mu\frac{d^2F'}{dt^2} \\ k\nabla^2 G' &= 4\pi\mu\frac{d^2G'}{dt^2} \\ k\nabla^2 H' &= 4\pi\mu\frac{d^2H'}{dt^2} \end{aligned}\right\}. \tag{78}$$

Hence the disturbances indicated by F', G', H' are propagated with the velocity $V=\sqrt{\dfrac{k}{4\pi\mu}}$ through the field; and since

$$\frac{dF'}{dx}+\frac{dG'}{dy}+\frac{dH'}{dz}=0,$$

the resultant of these disturbances is in the plane of the wave.

(99) The remaining part of the total disturbances F, G, H being the part depending on χ, is subject to no condition except that expressed in the equation

$$\frac{d\Psi}{dt}+\frac{d^2\chi}{dt^2}=0.$$

If we perform the operation ∇^2 on this equation, it becomes

$$ke=\frac{dJ}{dt}-k\nabla^2\phi(x, y, z). \tag{79}$$

Since the medium is a perfect insulator, e, the free electricity, is immoveable, and therefore $\dfrac{dJ}{dt}$ is a function of x, y, z, and the value of J is either constant or zero, or uniformly increasing or diminishing with the time; so that no disturbance depending on J can be propagated as a wave.

(100) The equations of the electromagnetic field, deduced from purely experimental evidence, show that transversal vibrations only can be propagated. If we were to go beyond our experimental knowledge and to assign a definite density to a substance which we should call the electric fluid, and select either vitreous or resinous electricity as the representative of that fluid, then we might have normal vibrations propagated with a velocity depending on this density. We have, however, no evidence as to the density of electricity, as we do not even know whether to consider vitreous electricity as a substance or as the absence of a substance.

Hence electromagnetic science leads to exactly the same conclusions as optical science with respect to the direction of the disturbances which can be propagated through the field; both affirm the propagation of transverse vibrations, and both give the same velocity of propagation. On the other hand, both sciences are at a loss when called on to affirm or deny the existence of normal vibrations.

Relation between the Index of Refraction and the Electromagnetic Character of the Substance.

(101) The velocity of light in a medium, according to the Undulatory Theory, is

$$\frac{1}{i} V_0,$$

where i is the index of refraction and V_0 is the velocity in vacuum. The velocity, according to the Electromagnetic Theory, is

$$\sqrt{\frac{k}{4\pi\mu}},$$

where, by equations (49) and (71), $k = \frac{1}{D} k_0$, and $k_0 = 4\pi V_0^2$.

Hence $$D = \frac{i^2}{\mu}, \qquad (80)$$

or the Specific Inductive Capacity is equal to the square of the index of refraction divided by the coefficient of magnetic induction.

Propagation of Electromagnetic Disturbances in a Crystallized Medium

(102) Let us now calculate the conditions of propagation of a plane wave in a medium for which the values of k and μ are different in different directions. As we do not propose to give a complete investigation of the question in the present imperfect state of the theory as extended to disturbances of short period, we shall assume that the axes of magnetic induction coincide in direction with those of electric elasticity.

(103) Let the values of the magnetic coefficient for the three axes

be λ, μ, ν, then thc equations of magnetic force (B) become

$$\left.\begin{aligned}\lambda\alpha &= \frac{dH}{dy}-\frac{dG}{dz}\\ \mu\beta &= \frac{dF}{dz}-\frac{dH}{dx}\\ \nu\gamma &= \frac{dG}{dx}-\frac{dF}{dy}\end{aligned}\right\}. \tag{81}$$

The equations of electric currents (C) remain as before.

The equations of electric elasticity (E) will be

$$\left.\begin{aligned}P &= 4\pi a^2 f\\ Q &= 4\pi b^2 g\\ R &- 4\pi c^2 h\end{aligned}\right\}, \tag{82}$$

where $4\pi a^2$, $4\pi b^2$, and $4\pi c^2$ are the values of k for the axes of x, y, z.

Combining these equations with (A) and (D), we get equations of the form

$$\frac{1}{\mu\nu}\left(\lambda\frac{d^2F}{dx^2}+\mu\frac{d^2F}{dy^2}+\nu\frac{d^2F}{dz^2}\right)-\frac{1}{\mu\nu}\frac{d}{dx}\left(\lambda\frac{dF}{dx}+\mu\frac{dG}{dy}+\nu\frac{dH}{dz}\right)$$
$$=\frac{1}{a^2}\left(\frac{d^2F}{dt^2}+\frac{d^2\Psi}{dxdt}\right). \tag{83}$$

(104) If l, m, n are the direction-cosines of the wave, and V its velocity, and if

$$lx+my+nz-Vt=w, \tag{84}$$

then F, G, H, and Ψ will be functions of w; and if we put F', G', H', Ψ' for the second differentials of these quantities with respect to w, the equations will be

$$\left.\begin{aligned}&\left\{V^2-a^2\left(\frac{m^2}{\nu}+\frac{n^2}{\mu}\right)\right\}F'+\frac{a^2lm}{\nu}G'+\frac{a^2ln}{\mu}H'-lV\Psi'=0\\ &\left\{V^2-b^2\left(\frac{n^2}{\lambda}+\frac{l^2}{\nu}\right)\right\}G'+\frac{b^2mn}{\lambda}H'+\frac{b^2ml}{\nu}F'-mV\Psi'=0\\ &\left\{V^2-c^2\left(\frac{l^2}{\mu}+\frac{m^2}{\lambda}\right)\right\}H'+\frac{c^2nl}{\mu}F'+\frac{c^2nm}{\lambda}G'-nV\Psi'=0\end{aligned}\right\}. \tag{85}$$

If we now put

$$V^4 - V^2\frac{1}{\lambda\mu\nu}\{l^2\lambda(b^2\mu+c^2\nu)+m^2\mu(c^2\nu+a^2\lambda)+n^2\nu(a^2\lambda+b^2\mu)\}$$
$$+\frac{a^2b^2c^2}{\lambda\mu\nu}\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)(l^2\lambda+m^2\mu+n^2\nu)=U \qquad (86)$$

we shall find $$F'V^2U-l\Psi'VU=0, \qquad (87)$$

with two similar equations for G' and H'. Hence either

$$V=0, \qquad (88)$$
$$U=0, \qquad (89)$$

or $$VF'=l\Psi', \qquad VG'=m\Psi' \quad \text{and} \quad VH'=n\Psi'. \qquad (90)$$

The third supposition indicates that the resultant of F', G', H' is in the direction normal to the plane of the wave; but the equations do not indicate that such a disturbance, if possible, could be propagated, as we have no other relation between Ψ' and F', G', H'.

The solution $V=0$ refers to a case in which there is no propagation.

*The solution $U=0$ gives two values for V^2 corresponding to

*[Although it is not expressly stated in the text it should be noticed that in finding equations (91) and (92) the quantity Ψ' is put equal to zero. See § 98 and also the corresponding treatment of this subject in the *Electricity and Magnetism*, ii. § 796. It may be observed that the equations referred to and the table given in § 105 may perhaps be more readily understood from a different mode of elimination. If we write

$$\lambda l^2+\mu m^2+\nu n^2=P\lambda\mu\nu \quad \text{and} \quad \lambda lF'+\mu mG'+\nu nH'=Q\lambda\mu\nu,$$

it is readily seen that $$F'=l\frac{V\Psi'-a^2\lambda Q}{V^2-a^2\lambda P},$$

with similar expressions for G', H'. From these we readily obtain by reasoning similar to that in § 104, the equation corresponding to (86), viz.:

$$\frac{l^2\lambda}{V^2-a^2\lambda P}+\frac{m^2\mu}{V^2-b^2\mu P}+\frac{n^2\nu}{V^2-c^2\nu P}=0.$$

This form of the equation agrees with that given in the *Electricity and Magnetism*, ii. § 797.

By means of this equation the equations (91) and (92) readily follow when $\Psi'=0$. The ratios of $F':G':H'$ for any direction of propagation may also be determined.]

values of F', G', H', which are given by the equations

$$\frac{l}{a^2}F'+\frac{m}{b^2}G'+\frac{n}{c^2}H'=0, \tag{91}$$

$$\frac{a^2l\lambda}{F'}(b^2\mu-c^2\nu)+\frac{b^2m\mu}{G'}(c^2\nu-a^2\lambda)+\frac{c^2n\nu}{H'}(a^2\lambda-b^2\mu)=0. \tag{92}$$

(105) The velocities along the axes are as follows:—

	Direction of propagation	x	y	z
Direction of the electric displacements	x		$\frac{a^2}{\nu}$	$\frac{a^2}{\mu}$
	y	$\frac{b^2}{\nu}$		$\frac{b^2}{\lambda}$
	z	$\frac{c^2}{\mu}$	$\frac{c^2}{\lambda}$	

Now we know that in each principal plane of a crystal the ray polarized in that plane obeys the ordinary law of refraction, and therefore its velocity is the same in whatever direction in that plane it is propagated.

If polarized light consists of electromagnetic disturbances in which the electric displacement is in the plane of polarization, then

$$a^2=b^2=c^2. \tag{93}$$

If, on the contrary, the electric displacements are perpendicular to the plane of polarization,

$$\lambda=\mu=\nu. \tag{94}$$

We know, from the magnetic experiments of Faraday, Plücker, &c., that in many crystals λ, μ, ν are unequal.

The experiments of Knoblauch* on electric induction through crystals seem to show that a, b and c may be different.

The inequality, however, of λ, μ, ν is so small that great magnetic

* *Philosophical Magazine*, 1852.

forces are required to indicate their difference, and the differences do not seem of sufficient magnitude to account for the double refraction of the crystals.

On the other hand, experiments on electric induction are liable to error on account of minute flaws, or portions of conducting matter in the crystal.

Further experiments on the magnetic and dielectric properties of crystals are required before we can decide whether the relation of these bodies to magnetic and electric forces is the same, when these forces are permanent as when they are alternating with the rapidity of the vibrations of light.

Relation between Electric Resistance and Transparency

(106) If the medium, instead of being a perfect insulator, is a conductor whose resistance per unit of volume is ρ, then there will be not only electric displacements, but true currents of conduction in which electrical energy is transformed into heat, and the undulation is thereby weakened. To determine the coefficient of absorption, let us investigate the propagation along the axis of x of the transverse disturbance G.

By the former equations

$$\frac{d^2G}{dx^2} = -4\pi\mu(q')$$

$$= -4\pi\mu\left(\frac{df}{dt} + q\right) \quad \text{by (A)},$$

$$\frac{d^2G}{dx^2} = +4\pi\mu\left(\frac{1}{k}\frac{d^2G}{dt^2} - \frac{1}{\rho}\frac{dG}{dt}\right) \quad \text{by (E) and (F).} \tag{95}$$

If G is of the form

$$G = e^{-px}\cos(qx + nt), \tag{96}$$

we find that

$$p = \frac{2\pi\mu}{\rho}\frac{n}{q} = \frac{2\pi\mu}{\rho}\frac{V}{i}, \tag{97}$$

where V is the velocity of light in air, and i is the index of refraction. The proportion of incident light transmitted through the thickness x is

$$e^{-2px}. \tag{98}$$

Let R be the resistance in electromagnetic measure of a plate of

the substance whose thickness is x, breadth b, and length l, then

$$R = \frac{l\rho}{bx},$$

$$2px = 4\pi\mu\frac{V}{i}\frac{l}{bR}. \tag{99}$$

(107) Most transparent solid bodies are good insulators, whereas all good conductors are very opaque.

Electrolytes allow a current to pass easily and yet are often very transparent. We may suppose, however, that in the rapidly alternating vibrations of light, the electromotive forces act for so short a time that they are unable to effect a complete separation between the particles in combination, so that when the force is reversed the particles oscillate into their former position without loss of energy.

Gold, silver, and platinum are good conductors, and yet when reduced to sufficiently thin plates they allow light to pass through them. If the resistance of gold is the same for electromotive forces of short period as for those with which we make experiments, the amount of light which passes through a piece of gold-leaf, of which the resistance was determined by Mr C. Hockin, would be only 10^{-50} of the incident light, a totally imperceptible quantity. I find that between $\frac{1}{500}$ and $\frac{1}{1000}$ of green light gets through such gold-leaf. Much of this is transmitted through holes and cracks; there is enough, however, transmitted through the gold itself to give a strong green hue to the transmitted light. This result cannot be reconciled with the electromagnetic theory of light, unless we suppose that there is less loss of energy when the electromotive forces are reversed with the rapidity of the vibrations of light than when they act for sensible times, as in our experiments.

Absolute Values of the Electromotive and Magnetic Forces called into play in the Propagation of Light

(108) If the equation of propagation of light is

$$F = A\cos\frac{2\pi}{\lambda}(z - Vt),$$

the electromotive force will be

$$P = -A\frac{2\pi}{\lambda}V\sin\frac{2\pi}{\lambda}(z - Vt);$$

and the energy per unit of volume will be

$$\frac{P^2}{8\pi\mu V^2},$$

where P represents the greatest value of the electromotive force. Half of this consists of magnetic and half of electric energy.

The energy passing through a unit of area is

$$W = \frac{P^2}{8\pi\mu V};$$

so that

$$P = \sqrt{8\pi\mu VW},$$

where V is the velocity of light, and W is the energy communicated to unit of area by the light in a second.

According to Pouillet's data, as calculated by Professor W. Thomson*, the mechanical value of direct sunlight at the Earth is

83·4 foot-pounds per second per square foot.

This gives the maximum value of P in direct sunlight at the Earth's distance from the Sun,

$$P = 60{,}000{,}000,$$

or about 600 Daniell's cells per metre.

At the Sun's surface the value of P would be about

13,000 Daniell's cells per metre.

At the Earth the maximum magnetic force would be ·193†.

At the Sun it would be 4·13.

These electromotive and magnetic forces must be conceived to be reversed twice in every vibration of light; that is, more than a thousand million million times in a second.

PART VII

CALCULATION OF THE COEFFICIENTS OF ELECTROMAGNETIC INDUCTION

General Methods

(109) The electromagnetic relations between two conducting circuits, A and B, depend upon a function M of their form and relative position, as has been already shown.

* *Transactions of the Royal Society of Edinburgh*, 1854 ("Mechanical Energies of the Solar System").

† The horizontal magnetic force at Kew is about 1·76 in metrical units.

M may be calculated in several different ways, which must of course all lead to the same result.

First Method. M is the electromagnetic momentum of the circuit B when A carries a unit current, or

$$M=\int\left(F\frac{dx}{ds'}+G\frac{dy}{ds'}+H\frac{dz}{ds'}\right)ds',$$

where F, G, H are the components of electromagnetic momentum due to a unit current in A, and ds' is an element of length of B, and the integration is performed round the circuit of B.

To find F, G, H, we observe that by (B) and (C)

$$\frac{d^2F}{dx^2}+\frac{d^2F}{dy^2}+\frac{d^2F}{dz^2}=-4\pi\mu p',$$

with corresponding equations for G and H, p', q', and r' being the components of the current in A.

Now if we consider only a single element ds of A, we shall have

$$p'=\frac{dx}{ds}ds,\qquad q'=\frac{dy}{ds}ds,\qquad r'=\frac{dz}{ds}ds,$$

and the solution of the equation gives

$$F=\frac{\mu}{\rho}\frac{dx}{ds}ds,\qquad G=\frac{\mu}{\rho}\frac{dy}{ds}ds,\qquad H=\frac{\mu}{\rho}\frac{dz}{ds}ds,$$

where ρ is the distance of any point from ds. Hence

$$\begin{aligned}M&=\int\int\frac{\mu}{\rho}\left(\frac{dx}{ds}\frac{dx}{ds'}+\frac{dy}{ds}\frac{dy}{ds'}+\frac{dz}{ds}\frac{dz}{ds'}\right)dsds'\\&=\int\int\frac{\mu}{\rho}\cos\theta dsds',\end{aligned}$$

where θ is the angle between the directions of the two elements ds, ds', and ρ is the distance between them, and the integration is performed round both circuits.

In this method we confine our attention during integration to the two linear circuits alone.

(110) Second Method. M is the number of lines of magnetic force which pass through the circuit B when A carries a unit current, or

$$M=\sum(\mu\alpha l+\mu\beta m+\mu\gamma n)\,dS',$$

where $\mu\alpha$, $\mu\beta$, $\mu\gamma$ are the components of magnetic induction due to unit current in A, S' is a surface bounded by the current B, and l, m, n are the direction-cosines of the normal to the surface, the integration being extended over the surface.

We may express this in the form

$$M = \mu \sum \frac{1}{\rho^2} \sin\theta \sin\theta' \sin\phi dS' ds,$$

where dS' is an element of the surface bounded by B, ds is an element of the circuit A, ρ is the distance between then, θ and θ' are the angles between ρ and ds and between ρ and the normal to dS' respectively, and ϕ is the angle between the planes in which θ and θ' are measured. The integration is performed round the circuit A and over the surface bounded by B.

This method is most convenient in the case of circuits lying in one plane, in which case $\sin\theta = 1$, and $\sin\phi = 1$.

(111) Third Method. M is that part of the intrinsic magnetic energy of the whole field which depends on the product of the currents in the two circuits, each current being unity.

Let α, β, γ be the components of magnetic intensity at any point due to the first circuit, α', β', γ' the same for the second circuit; then the intrinsic energy of the element of volume dV of the field is

$$\frac{\mu}{8\pi}\{(\alpha+\alpha')^2+(\beta+\beta')^2+(\gamma+\gamma')^2\}\, dV.$$

The part which depends on the product of the currents is

$$\frac{\mu}{4\pi}(\alpha\alpha'+\beta\beta'+\gamma\gamma')\, dV.$$

Hence if we know the magnetic intensities I and I' due to the unit current in each circuit, we may obtain M by integrating

$$\frac{\mu}{4\pi}\sum II' \cos\theta\, dV$$

over all space, where θ is the angle between the directions of I and I'.

Application to a Coil

(112) To find the coefficient (M) of mutual induction between two circular linear conductors in parallel planes, the distance be-

tween the curves being everywhere the same, and small compared with the radius of either.

If r be the distance between the curves, and a the radius of either, then when r is very small compared with a, we find by the second method, as a first approximation,

$$M = 4\pi a\left(\log_e \frac{8a}{r} - 2\right).$$

To approximate more closely to the value of M, let a and a_1 be the radii of the circles, and b the distance between their planes; then

$$r^2 = (a - a_1)^2 + b^2.$$

We obtain M by considering the following conditions:—

1st. M must fulfil the differential equation

$$\frac{d^2M}{da^2} + \frac{d^2M}{db^2} + \frac{1}{a}\frac{dM}{da} = 0.$$

This equation being true for any magnetic field symmetrical with respect to the common axis of the circles, cannot of itself lead to the determination of M as a function of a, a_1, and b. We therefore make use of other conditions.

2ndly. The value of M must remain the same when a and a_1 are exchanged.

3rdly. The first two terms of M must be the same as those given above.

M may thus be expanded in the following series:—

$$\begin{aligned} M = 4\pi a \log\frac{8a}{r}&\left\{1 + \frac{1}{2}\frac{a - a_1}{a} + \frac{1}{16}\frac{3b^2 + (a_1 - a)^2}{a^2}\right. \\ &\left. - \frac{1}{32}\frac{\{3b^2 + (a - a_1)^2\}(a - a_1)}{a^3} + \&c.\right\} \\ &- 4\pi a\left[2 + \frac{1}{2}\frac{a - a_1}{a} + \frac{1}{16}\frac{b^2 - 3(a - a_1)^2}{a^2}\right. \\ &\left. - \frac{1}{48}\frac{\{6b^2 - (a - a_1)^2\}(a - a_1)}{a^3} + \&c.\right]. \end{aligned}$$

(113) We may apply this result to find the coefficient of self-induction (L) of a circular coil of wire whose section is small compared with the radius of the circle.

Let the section of the coil be a rectangle, the breadth in the plane

of the circle being c, and the depth perpendicular to the plane of the circle being b.

Let the mean radius of the coil be a, and the number of windings n: then we find, by integrating,

$$L = \frac{n^2}{b^2c^2}\iiiint M(xy\,x'\,y')\,dx\,dy\,dx'\,dy',$$

where $M(xy\,x'y')$ means the value of M for the two windings whose coordinates are xy and $x'y'$ respectively; and the integration is performed first with respect to x and y over the rectangular section, and then with respect to x' and y' over the same space.

$$\begin{aligned}
L = 4\pi n^2 a\Big\{&\log_e\frac{8a}{r}+\frac{1}{12}-\frac{4}{3}\left(\theta-\frac{\pi}{4}\right)\cot 2\theta-\frac{\pi}{3}\cos 2\theta \\
&-\frac{1}{6}\cot^2\theta\log\cos\theta-\frac{1}{6}\tan^2\theta\log\sin\theta\Big\} \\
&+\frac{\pi n^2 r^2}{24a}\Big\{\log\frac{8a}{r}(2\sin^2\theta+1)+3{\cdot}45+27{\cdot}475\cos^2\theta \\
&\quad 3{\cdot}2\left(\frac{\pi}{2}-\theta\right)\frac{\sin^3\theta}{\cos\theta}+\frac{1}{5}\frac{\cos^4\theta}{\sin^2\theta}\log\cos\theta \\
&+\frac{13}{3}\frac{\sin^4\theta}{\cos^2\theta}\log\sin\theta\Big\}+\&c.
\end{aligned}$$

Here a = mean radius of the coil.
Here r = diagonal of the rectangular section $= \sqrt{b^2+c^2}$.
Here θ = angle between r and the plane of the circle.
Here n = number of windings.

The logarithms are Napierian, and the angles are in circular measure.

In the experiments made by the Committee of the British Association for determining a standard of Electrical Resistance, a double coil was used, consisting of two nearly equal coils of rectangular section, placed parallel to each other, with a small interval between them.

The value of L for this coil was found in the following way.

The value of L was calculated by the preceding formula for six different cases, in which the rectangular section considered has

always the same breadth, while the depth was

$$A,\ B,\ C, \qquad A+B, \qquad B+C, \qquad A+B+C,$$

and $n=1$ in each case.

Calling the results

$$L(A), \qquad L(B), \qquad L(C), \&c.,$$

we calculate the coefficient of mutual induction $M(AC)$ of the two coils thus,

$$2ACM(AC)=(A+B+C)^2L(A+B+C)-(A+B)^2L(A+B) \\ -(B+C)^2L(B+C)+B^2L(B).$$

Then if n_1 is the number of windings in the coil A and n_2 in the coil C, the coefficient of self-induction of the two coils together is

$$L=n_1{}^2L(A)+2n_1n_2M(AC)+n_2{}^2L(C).$$

(114) These values of L are calculated on the supposition that the windings of the wire are evenly distributed so as to fill up exactly the whole section. This, however, is not the case, as the wire is generally circular and covered with insulating material. Hence the current in the wire is more concentrated than it would have been if it had been distributed uniformly over the section, and the currents in the neighbouring wires do not act on it exactly as such a uniform current would do.

The corrections arising from these considerations may be expressed as numerical quantities, by which we must multiply the length of the wire, and they are the same whatever be the form of the coil.

Let the distance between each wire and the next, on the supposition that they are arranged in square order, be D, and let the diameter of the wire be d, then the correction for diameter of wire is

$$+2\left(\log\frac{D}{d}+\frac{4}{3}\log 2+\frac{\pi}{3}-\frac{11}{6}\right).$$

The correction for the eight nearest wires is

$$+0\cdot0236.$$

For the sixteen in the next row

$$+0\cdot00083.$$

These corrections being multiplied by the length of wire and added to the former result, give the true value of L, considered as

the measure of the potential of the coil on itself for unit current in the wire when that current has been established for some time, and is uniformly distributed through the section of the wire.

(115) But at the commencement of a current and during its variation the current is not uniform throughout the section of the wire, because the inductive action between different portions of the current tends to make the current stronger at one part of the section than at another. When a uniform electromotive force P arising from any cause acts on a cylindrical wire of specific resistance ρ, we have

$$p\rho = P - \frac{dF}{dt},$$

where F is got from the equation

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = -4\pi\mu p,$$

r being the distance from the axis of the cylinder.

Let one term of the value of F be of the form Tr^n, where T is a function of the time, then the term of p which produced it is of the form

$$-\frac{1}{4\pi\mu} n^2 T r^{n-2}.$$

Hence if we write

$$F = T + \frac{\mu\pi}{\rho}\left(-P + \frac{dT}{dt}\right)r^2 + \left(\frac{\mu\pi}{\rho}\right)^2 \frac{1}{1^2 . 2^2}\frac{d^2T}{dt^2} r^4 + \&c.$$

$$p\rho = \left(P - \frac{dT}{dt}\right) - \frac{\mu\pi}{\rho}\frac{d^2T}{dt^2} r^2 - \left(\frac{\mu\pi}{\rho}\right)^2 \frac{1}{1^2 . 2^2}\frac{d^3T}{dt^3} r^4 - \&c.$$

The total counter current of self-induction at any point is

$$\int\left(\frac{P}{\rho} - p\right)dt = \frac{1}{\rho}T + \frac{\mu\pi}{\rho^2}\frac{dT}{dt}r^2 + \frac{\mu^2\pi^2}{\rho^3}\frac{1}{1^2 2^2}\frac{d^2T}{dt^2}r^4 + \&c.$$

from $t = 0$ to $t = \infty$.

When $t = 0$, $p = 0$,

$$\therefore \left(\frac{dT}{dt}\right) = P, \qquad \left(\frac{d^2T}{dt^2}\right)_0 = 0, \&c.$$

When $t=\infty$, $p=\frac{P}{\rho}$,

$$\therefore \left(\frac{dT}{dt}\right)_\infty = 0, \qquad \left(\frac{d^2T}{dt^2}\right)_\infty = 0, \text{ \&c.}$$

$$\int_0^\infty \int_0^r 2\pi\left(\frac{P}{\rho}-p\right) r\,dr\,dt = \frac{1}{\rho}\,T\pi r^2 + \frac{1}{2}\frac{\mu\pi^2}{\rho^2}\frac{dT}{dt}r^4 + \frac{\mu^2\pi^3}{\rho^3}\frac{1}{1^2.2^2.3}\frac{d^2T}{dt^2}r^6 + \text{\&c.}$$

from $t=0$ to $=\infty$.

When $t=0$, $p=0$ throughout the section,

$$\therefore \left(\frac{dT}{dt}\right)_0 = P, \qquad \left(\frac{d^2T}{dt^2}\right)_0 = 0, \text{ \&c.}$$

When $t=\infty$, $p=0$ throughout the section,

$$\therefore \left(\frac{dT}{dt}\right)_\infty = 0, \qquad \left(\frac{d^2T}{dt^2}\right)_\infty = 0, \text{ \&c.}$$

Also if l be the length of the wire, and R its resistance,

$$R=\frac{\rho l}{\pi r^2};$$

and if C be the current when established in the wire, $C=\frac{Pl}{R}$.

The total counter current may be written

$$\frac{l}{R}(T_\infty - T_0) - \tfrac{1}{2}\mu\frac{l}{R}C = -\frac{LC}{R} \text{ by § (35).}$$

Now if the current instead of being variable from the centre to the circumference of the section of the wire had been the same throughout, the value of F would have been

$$F = T + \mu\gamma\left(1-\frac{r^2}{r_0^2}\right),$$

where γ is the current in the wire at any instant, and the total counter current would have been

$$\int_0^\infty \int_0^r \frac{1}{\rho}\frac{dF}{dt} 2\pi r\,dr = \frac{l}{R}(T_\infty - T_0) - \tfrac{3}{4}\mu\frac{l}{R}C = -\frac{L'C}{R}, \quad \text{say.}$$

Hence $$L = L' - \tfrac{1}{4}\mu l,$$

or the value of L which must be used in calculating the self-induction of a wire for variable currents is less than that which is deduced from the supposition of the current being constant throughout the section of the wire by $\frac{1}{3}\mu l$, where l is the length of the wire, and μ is the coefficient of magnetic induction for the substance of the wire.

(116) The dimensions of the coil used by the Committee of the British Association in their experiments at King's College in 1864 were as follows:

		metre.
Mean radius	$= a =$	·158194
Depth of each coil	$= b =$	·01608
Breadth of each coil	$= c =$	·01841
Distance between the coils	$=$	·02010
Number of windings	$n =$	313
Diameter of wire	$=$	·00126

The value of L derived from the first term of the expression is 437440 metres.

The correction depending on the radius not being infinitely great compared with the section of the coil as found from the second term is −7345 metres.

The correction depending on the diameter of the wire is per unit of length	+·44997
Correction of eight neighbouring wires	+·0236
For sixteen wires next to these	+·0008
Correction for variation of current in different parts of section	−·2500
Total correction per unit of length	·22437
Length	311·236 metres.
Sum of corrections of this kind	70 metres.
Final value of L by calculation	430165 metres.

This value of L was employed in reducing the observations, according to the method explained in the Report of the Committee*. The correction depending on L varies as the square of the velocity. The results of sixteen experiments to which this correction had been applied, and in which the velocity varied from 100 revolutions in seventeen seconds to 100 in seventy-seven seconds,

* *British Association Reports*, 1863, p. 169.

were compared by the method of least squares to determine what further correction depending on the square of the velocity should be applied to make the outstanding errors a minimum.

The result of this examination showed that the calculated value of L should be multiplied by 1·0618 to obtain the value of L, which would give the most consistent results.

We have therefore L by calculation	430165 metres.
Probable value of L by method of least squares	456748 metres.
Result of rough experiment with the Electric Balance (see § 46)	410000 metres.

The value of L calculated from the dimensions of the coil is probable much more accurate than either of the other determinations.

Thomas F. Torrance was born in China in 1913 to missionary parents. For nearly 30 years until his retirement in 1979 he was Professor of Christian Dogmatics ("Systematic Theology") at the University of Edinburgh, Scotland. Amongst many other topics, he has published numerous books and papers on the "Philosophy of Theology" dealing with the epistemology of theological concepts and with the relation of theology to natural science. In 1976-7 he served as Moderator of the General Assembly of the Church of Scotland, and in 1978 he was awarded the Templeton Prize for Progress in Religion. Having reached the 21st Century, he is still an active scholar and researcher.

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